Homework 11

Math 561: Theory of Probability I

Due date: April 20, 2023

Each problem is worth 10 points and only five randomly chosen problems will be graded if there are more than 5 problems. Please indicate whom you worked with, it will not affect your grade in any way.

1. Let X, Y be random variables, and suppose Y is measurable with respect to some sub-σ-field $\mathcal{G}$. Let $Q(\omega, \cdot)$ be a regular conditional distribution for X given $\mathcal{G}$. Prove that, for bounded measurable h,

$$\mathbb{E}(h(X,Y) \mid \mathcal{G})(\omega) = \int h(x, Y(\omega))Q(\omega, dx) \text{ a.s.}$$

2. For $i = 1, 2$, let X_i be a r.v. defined on $(\Omega, \mathcal{F}, \mathbb{P})$ taking values in $(S_i, \mathcal{S}_i)$. Let $\mathcal{G}$ be a sub-σ-field of $\mathcal{F}$. Prove that assertions (a),(b) and (c) below are equivalent. When these assertions hold, we say that X_1 and X_2 are **conditionally independent given** $\mathcal{G}$.

   (a) $\mathbb{P}(X_1 \in A_1, X_2 \in A_2 \mid \mathcal{G}) = \mathbb{P}(X_1 \in A_1 \mid \mathcal{G})\, \mathbb{P}(X_2 \in A_2 \mid \mathcal{G})$ for all $A_i \in \mathcal{S}_i$.

   (b) $\mathbb{E}(h_1(X_1)h_2(X_2) \mid \mathcal{G}) = \mathbb{E}(h_1(X_1) \mid \mathcal{G})\, \mathbb{E}(h_2(X_2) \mid \mathcal{G})$ for all bounded measurable $h_i : S_i \to \mathbb{R}$.

   (c) $\mathbb{E}(h_1(X_1) \mid \mathcal{G}, X_2) = \mathbb{E}(h_1(X_1) \mid \mathcal{G})$ for all bounded measurable $h_1 : S_1 \to \mathbb{R}$. Here $\mathbb{E}(\cdot \mid \mathcal{G}, X_2)$ means $\mathbb{E}(\cdot \mid \sigma(\mathcal{G}, \sigma(X_2)))$.

3. Suppose X and Y are conditionally independent given Z. Suppose X and Z are conditionally independent given $\mathcal{F}$, where $\mathcal{F} \subseteq \sigma(Z)$. Prove that X and Y are conditionally independent given $\mathcal{F}$.
   Hint: use the previous problem.

4. Let (M_n) be a sub-martingale w.r.t the filtration $(\mathcal{F}_n)$.
   (a) Prove that, (M_n) is a sub-martingale w.r.t. $(\mathcal{G}_n)$ where $\mathcal{G}_n = \sigma(M_1, M_2, \ldots, M_n), n \geqslant 1$.
   (b) Let (Q_n) be another sub-martingales w.r.t. $(\mathcal{F}_n)$. Show that $(M_n + Q_n)$ and that $(\max(M_n, Q_n))$ are also sub-martingales w.r.t. $(\mathcal{F}_n)$.
   (c) Give an example where

   (M_n) is a sub-martingale w.r.t. $(\mathcal{F}_n)$,
   (Q_n) is a sub-martingale w.r.t. some other filtration $(\mathcal{G}_n)$,
   but $(M_n + Q_n)$ is NOT a sub-martingale w.r.t. any filtration.

5. Let S, T be two stopping times w.r.t. the same filtration $(\mathcal{F}_n, n \geqslant 0)$. Prove or disprove whether the following are stopping times: (i) $\max\{S, T\}$ (ii) $\min\{S, T\}$ (iii) $|S - T|$
   (iv) $\phi(S)$ where $\phi : \{0, 1, 2, \ldots\} \to \{0, 1, 2, \ldots\}$ is non-decreasing with $\phi(x) \geqslant x$ for all x.