Homework 1

MATH 561/STAT 551: Theory of Probability I

Due date: January 26, 2023

Each problem is worth 10 points and only **five randomly chosen problems** will be graded. Please indicate whom you worked with, it will not affect your grade in any way. The link to upload the solution is available in canvas.

1. **(Size of a σ-field.)** Suppose that $\mathcal{F}$ is a σ-field of subsets of some sample space Ω (Ω need not be finite). Assume that there are only finitely many sets which belong to this σ-field. This question is about how many sets could there be in $\mathcal{F}$? Can it be 3 or 10 or 16? Characterize the set of integers that can appear as the size of a finite σ-field.

2. **(Increasing limits of fields.)** Let $(\mathcal{F}_n)_{n\geqslant 1}$ be a strictly increasing sequence of fields, *i.e.,* $\mathcal{F}_n \subseteq \mathcal{F}_{n+1}, \mathcal{F}_n \neq \mathcal{F}_{n+1}$ for $n = 1, 2, \ldots$. Define $\mathcal{F} = \bigcup_{n=1}^{\infty} \mathcal{F}_n$.

   (a) Show that $\mathcal{F}$ is a field.
   (b) Give an example to show that, if each $\mathcal{F}_n$ is a σ-field, then $\mathcal{F}$ need not be a σ-field.

   In fact, $\bigcup_{n=1}^{\infty} \mathcal{F}_n$ is never a σ-field, when $\mathcal{F}_n$'s are strictly increasing (no need to prove this).

3. **(Countable generator.)** Suppose $B \in \sigma(\mathcal{C})$, for some collection $\mathcal{C}$ of subsets. Show there exists a **countable subcollection** $\mathcal{C}_B$ of $\mathcal{C}$ such that $B \in \sigma(\mathcal{C}_B)$.

4. **(Characterizing generated fields.)** Given a non-empty collection $\mathcal{C}$ of sets, recall that $\mathcal{A}(\mathcal{C})$ is the intersection of all fields containing $\mathcal{C}$. Show that $\mathcal{A}(\mathcal{C})$ is the class of sets of the form

   $$\cup_{i=1}^{m} \cap_{j=1}^{n_i} A_{i,j}$$

   where for each i, j either $A_{i,j} \in \mathcal{C}$ or $A_{i,j}^c \in \mathcal{C}$, and where the m sets $\cap_{j=1}^{n_i} A_{i,j}, i = 1, 2, \ldots, m$ are disjoint.

5. Give an example of a measurable space $(\Omega, \mathcal{F})$, a collection $\mathcal{A}$ and probability measures μ and ν such that

   i) $\mu(A) = \nu(A)$ for all $A \in \mathcal{A}$.
   ii) $\mathcal{F} = \sigma(\mathcal{A})$.
   iii) $\mu \neq \nu$.

   Hint: Start with $\Omega = \{1, 2, 3, 4\}$.

6. **(Approximating σ-field by field.)** Let μ be a probability measure on $(\Omega, \mathcal{F})$, where $\mathcal{F} = \sigma(\mathcal{A})$ for a field $\mathcal{A}$. Show that for each $B \in \mathcal{F}$ and $\varepsilon > 0$ there exists $A \in \mathcal{A}$ such that $\mu(B \triangle A) < \varepsilon$. Here $B \triangle A = (B \setminus A) \cup (A \setminus B)$ is the symmetric difference between A, B.

   Hint: Consider the collection of all B satisfying this property.