1. (20 points) Suppose that $X_1, X_2, \ldots$ are uncorrelated random variables with $E(X_j) = \mu_j$ and $\text{Var}(X_j) = \sigma_j^2$ (i.e., $E(X_i - \mu_i)(X_j - \mu_j) = 0$ for $i \neq j$). Let $S_n = X_1 + \cdots + X_n$. Assume that $\mu_j \to 0$ and $\sigma_j^2/j \to 0$ as $j \to \infty$. Show that as $n \to \infty$, 

$$\frac{S_n}{n} \to 0 \quad \text{in probability}.$$
2. (5+30+5 points) Suppose that \(X_1, X_2, \ldots\) are uncorrelated random variables. Let \(S_n = X_1 + \cdots + X_n\). Assume that \(P(0 \leq X_i \leq 1 \text{ for all } i \geq 1) = 1\) and \(E(S_n) \to \infty\) as \(n \to \infty\).

(a) Show that \(\text{Var}(S_n) \leq E(S_n)\) for all \(n \geq 1\).

(b) Use the Borel-Cantelli lemma to prove that

\[
\frac{S_n}{E(S_n)} \to 1 \text{ almost surely.}
\]

(Hint: First prove convergence for an appropriately chosen subsequence.)

(c) Use the above result to argue that, if \(A_1, A_2, \ldots\) are pairwise independent events with \(\sum_{n \geq 1} P(A_n) = \infty\), then

\[
P(A_n \text{ infinitely often}) = 1.
\]