Homework 3
Math 561: Theory of Probability I
Due date: February 7, 2018

Each problem is worth 10 points and only five randomly chosen problems will be graded if there are more than 5 problems. Please indicate whom you worked with, it will not affect your grade in any way.

1. Prove that the $E$ defined on the probability space $(\Omega, \mathcal{F}, P)$ using the four-step procedure satisfies

$$E(X + Y) = E X + E Y$$

and $E(cX) = cE X$ for all integrable r.v. $X, Y$ and $c \in \mathbb{R}$.

**Solution:** We have to check the conclusion at all four steps in the definition of the expectation.

Step 1. (Simple r.v.) For $\phi = \sum_{i=1}^n a_i \mathbb{1}_{A_i}$, $\psi = \sum_{j=1}^m b_j \mathbb{1}_{B_j}$ where $A_i$’s are disjoint, $B_j$’s are disjoint and $a_i \neq 0, b_j \neq 0$ for all $i, j$. We define $A_0 = (\cup_{i=1}^n A_i)^c, B_0 = (\cup_{j=1}^m B_j)^c, a_0 = b_0 = 0$. Then we have

$$\phi + \psi = \sum_{i=0}^n \sum_{j=0}^m (a_i + b_j) \mathbb{1}_{A_i B_j}$$

and $A_i B_j, 0 \leq i \leq n, 0 \leq j \leq m$ are mutually disjoint. From here we can easily get

$$E \phi + E \psi = E(\phi + \psi).$$

The part $E(c\phi) = cE \phi$ follows easily. Proof of the next three parts follow in a similar way.

Step 2. (Nonnegative bounded rvs.)

Step 3. (Nonnegative rvs.)

Step 4. (Integrable rvs.)

2. Let $X : ([0, 1], \mathcal{B}[0, 1], P = \text{Lebesgue measure}) \to (\mathbb{R}, \mathcal{B})$ be an integrable random variable. Given $\varepsilon > 0$, show that there exists a continuous function $Y : [0, 1] \to \mathbb{R}$ such that $E |X - Y| \leq \varepsilon$.

**Solution:** First we approximate $X$ by simple functions with bounded intervals.

Fix $\varepsilon > 0$. By definition of $E$, there exists a simple function $\phi = \sum_{i=1}^k a_i \mathbb{1}_{A_i}, a_i \neq 0$ for all $i$, $A_i$’s are disjoint subsets of $[0, 1]$ s.t. $E |X - \phi| \leq \varepsilon/4$. For every $i = 1, 2, \ldots, k$, we can find (by HW1 question 7.) a disjoint union of intervals of the form $(a, b]$, say $B_i$, such that $P(A_i \Delta B_i) \leq \varepsilon/(4 \sum_{i=1}^k |a_i|)$. W.L.O.G. suppose each $B_i$ is of the form $(a_i, b_i]$. Thus for $\psi(x) = \sum_{i=1}^k a_i \mathbb{1}_{B_i}(x)$ we have $E |\phi - \psi| \leq \varepsilon/4$.

Finally, take a continuous function $g_i : [0, 1] \to [0, 1]$ such that $g \equiv 1$ on $[a_i, b_i]$ and $\equiv 0$ on $[a_i - \delta, b_i + \delta]\varepsilon$ (e.g., do a piece-wise linear approximation). Choose $\delta = \varepsilon/(4 \sum_{i=1}^k |a_i|)$ and $Y = \sum_{i=1}^k a_i g_i$ to complete the proof.

3. Consider the probability space $(\mathcal{B}, \mathcal{F}, P)$, where $P$ has density $f(x)$, i.e.,

$$P(A) = \int_A f(x)dx$$

for a measurable function $f$. Show that, for an integrable r.v. $X$ on $(\mathcal{B}, \mathcal{F}, P)$ we have

$$E X = \int_{-\infty}^{\infty} X(x) f(x)dx.$$

**Solution:** We use the same four-step procedure used in the definition of expectation. For every $A \in \mathcal{B}$ we have $P(A) = \int_A f(x)dx$. Thus for any simple function $\phi = \sum_{i=1}^k a_i \mathbb{1}_{A_i}$, we have

$$E \phi = \sum_{i=1}^k a_i P(A_i) = \sum_{i=1}^k a_i \int_{A_i} f(x)dx = \sum_{i=1}^k a_i \int_{A_i} \mathbb{1}_{A_i}(x) \cdot f(x)dx = \int \phi(x) f(x)dx.$$
4. Let $X$ be a non-negative bounded r.v. on $(\mathbb{R}, \mathcal{B}, \mathbb{P})$. By definition, for a sequence of simple functions $\phi_n \uparrow h$ we have
\[
Eh = \lim_{n \to \infty} E\phi_n = \lim_{n \to \infty} \int \phi_n(x)f(x)dx = \int h(x)f(x)dx,
\]
where the last equality follows by $0 \leq \phi_n \leq h \leq M$ for some constant $M \in (0, \infty)$.

Similarly, for any non-negative r.v. $X$ we can get

(a) Show that, $\lim_{n \to \infty} \mathbb{P}(X_n - X) = 0$.
(b) Show that, if $\lim_{n \to \infty} \mathbb{P}(X_n - X) = 0$, then $X_n \to X$ in probability.
(c) Show that the converse is not true, i.e., give an example where $X_n \to X$ in probability but $X_n \not\to X$ a.s.

**Solution:**

(a) Take a sequence of positive real numbers $a_n$ such that $a_n \uparrow \infty$ as $n \to \infty$, but $a_{n+1} - a_n \to 0$. Consider the probability space $\Omega = ((0, 1), \mathcal{B}, \lambda)$ and the sequence of subsets $A_n = (a_n, a_{n+1})$ mod 1 (e.g., $0, 1$). Define the r.v.s $X_n := 1_{A_n}$. We claim that $X_n \to 0$ in probability. This is because $\mathbb{P}(A_n) = a_{n+1} - a_n \to 0$. However, $\{X_n\}$ converges to 0. Thus $X_n \not\to X$ a.s.

(b) Take a sequence of positive real numbers $a_n, n \geq 1$ such that $a_n \uparrow \infty$ as $n \to \infty$, but $a_{n+1} - a_n \to 0$. Consider the probability space $\Omega = ((0, \infty), \mathcal{B}, \lambda)$ and the sequence of subsets $A_n = (a_n, a_{n+1})$ mod 1 (e.g., $(0, 1, 2, 3, 4)$). Define the r.v.s $X_n := 1_{A_n}$. We claim that $X_n \to 0$ in probability. This is because $\mathbb{P}(A_n) = a_{n+1} - a_n \to 0$. However, $\{X_n\}$ converges to 0. Thus $X_n \not\to X$ a.s.

5. Use the monotone convergence theorem to prove the following.

(i) If $X_n \geq 0, X_n \uparrow X$ a.s. and $E(X_n) < \infty$ for some $n$ then $E(X_n) \downarrow E(X)$.
(ii) If $E|X| < \infty$ then $E(|X|1_{|X| > n}) \to 0$ as $n \to \infty$.
(iii) If $E(|X|) < \infty$ and $X_n \uparrow X$ a.s. then $E(X_n) \uparrow E(X)$ or else $E(X_n) \uparrow \infty$ and $E(|X|) = \infty$.
(iv) If $X$ takes values in the non-negative integers then
\[
E(X) = \sum_{n=1}^\infty \mathbb{P}(X \geq n).
\]

**Solution:**

(i) W.L.O.G. assume that $E(X_1) < \infty$. Define $Y_n := X_1 - X_n$. Clearly, $E(Y_n) < \infty$, $Y_n \geq 0$ and $Y_n \uparrow X_1 - X$. By MCT, $E(X_1 - X_n) \uparrow E(X_1 - X)$ and thus $E(X_n) \downarrow E(X)$.

(ii) Clearly, $|X|1_{|X| \leq n} \uparrow |X|$. By MCT, $E(|X|1_{|X| \leq n}) \uparrow E(|X|)$ or $E(|X|1_{|X| > n}) \downarrow 0$.

(iii) Clearly, $X_n^+ \uparrow X^+$ and $X_n^- \downarrow X^-$. Note that $E(X^-) \leq E(X_n^-) < \infty$ and by part (i) we have $E(X_n^-) \downarrow E(X^-)$.

We now separate into two cases: (a) $E(X^+) < \infty$ and (b) $E(X^+) = \infty$. In the first case by MCT, $E(X_n^+) \uparrow E(X^+)$ and we are done.

In the second case, given any $K > 0$ show that there exist $N$ such that for all $n \geq N$ we have $E(X_n^+) \geq E(X_n^+1_{X_n^+ < n}) \geq K$.

(iv) Write $\mathbb{P}(X \geq n) = \sum_{k \geq n} \mathbb{P}(X = k)$ and rearrange the sum.

6. Let $X$ be a r.v. with $||X||_p < \infty$. Prove that $||X||_p \to ||X||_\infty$ as $p \to \infty$.

**Solution:** It is easy to check that, $||X||_p \leq ||X||_\infty$. Let $||X||_\infty = t$. Fix $\varepsilon > 0$. By definition, $p_\varepsilon := \mathbb{P}(X \geq t - \varepsilon) > 0$. Thus $E|X|^p \geq (t - \varepsilon)p_\varepsilon$ or $||X||_p \geq (t - \varepsilon)p_\varepsilon^{1/p}$. Taking $p \to \infty$, we have $\lim_{p \to \infty} ||X||_p \geq t - \varepsilon$. This completes the proof.

7. Prove that $L^1(\Omega, \mathcal{F}, \mathbb{P}) = \{X \text{ r.v. on } (\Omega, \mathcal{F}, \mathbb{P}) \text{ with } E|X| < \infty\}$ is a Banach space (complete normed vector space) under the norm $||X|| = E|X|$.

**Solution:** We will prove the following:

(a) $L^1(\Omega, \mathcal{F}, \mathbb{P})$ is a vector space.

(b) $|| \cdot ||$ is a norm.
Proof of part (a) follows from the fact that $|X + Y| \leq |X| + |Y|$ and by linearity of expectation. For part (b), we also use the fact $X \geq 0$ implies that $E X \geq 0$ and $E |X| = 0$ implies $X = 0$ a.s. The only non-trivial part is proof of part (c).

Let $\phi_n$ be a Cauchy sequence in $L^1(\Omega, \mathcal{F}, \mathbb{P})$ w.r.t. $\|\|$. Thus, for every $\varepsilon > 0$ there exists $N(\varepsilon) \geq 1$ such that

$$\|\phi_n - \phi_m\| \leq \varepsilon$$

for all $n, m \geq N(\varepsilon)$.

We define $n_i = N(2^{-i}), i \geq 1$. W.l.o.g. we can assume that $n_1 < n_2 < \ldots$. Thus we have $E \left| \phi_{n_{i+1}} - \phi_{n_i} \right| \leq 2^{-i}$ for all $i$. In particular, by MCT we get that the r.v. $|\phi_{n_1}| + \sum_{i=1}^{\infty} \left| \phi_{n_{i+1}} - \phi_{n_i} \right|$ is in $L^1(\Omega, \mathcal{F}, \mathbb{P})$ and hence the r.v. $\psi := \phi_{n_1} + \sum_{i=1}^{\infty} (\phi_{n_{i+1}} - \phi_{n_i}) = \lim_{i \to \infty} \phi_{n_i}$ exists a.s. and is also in $L^1(\Omega, \mathcal{F}, \mathbb{P})$. It is now easy to check by DCT that $\|\psi - \phi_n\| \to 0$ as $n \to \infty$. The missing steps are easy to fill in.