Homework 2
Math 561/Stat 551: Theory of Probability I
Due date: January 31, 2018

Each problem is worth 10 points and only five randomly chosen problems will be graded if there are more than 5 problems. Please indicate whom you worked with, it will not affect your grade in any way.

1. Show that a distribution function $F$ has at most countably many discontinuities, i.e., the set $\{ x \mid F \text{ is discontinuous at } x \}$ is countable.

2. (Patching two r.v.s) Suppose $X, Y$ are two random variables on $(\Omega, \mathcal{F}, P)$ and let $A \in \mathcal{F}$. Show that the function

$$
Z(\omega) = \begin{cases} 
X(\omega) & \text{if } \omega \in A, \\
Y(\omega) & \text{if } \omega \in A^c
\end{cases}
$$

is a random variable.

3. Show that the Borel $\sigma$-field on $\mathbb{R}^d$ is the smallest $\sigma$-field that makes all continuous functions $f : \mathbb{R}^d \to \mathbb{R}$ measurable.

4. A function $f : \mathbb{R}^d \to \mathbb{R}$ is lower semicontinuous (l.s.c.) if $\liminf_{y \to x} f(y) \geq f(x)$ for all $x$. A function is upper semicontinuous (u.s.c.) if $\limsup_{y \to x} f(y) \leq f(x)$ for all $x$. Show that, if $f$ is l.s.c. or u.s.c., then $f$ is measurable.

5. Let $F : \mathbb{R}^2 \to [0, \infty)$ be a function satisfying the following properties:
   (i) $F$ is non-decreasing, i.e., $x_1 \leq x_2, y_1 \leq y_2$ implies that $F(x_1, y_1) \leq F(x_2, y_2)$,
   (ii) $F$ is right continuous, i.e., $\lim_{x_n \to x, y_n \to y} F(x_n, y_n) = F(x, y)$ and
   (iii) for every $x_1 \leq x_2, y_1 \leq y_2$ we have

$$
F(x_2, y_2) - F(x_2, y_1) - F(x_1, y_2) + F(x_1, y_1) \geq 0.
$$

Show that there exists a unique measure $\mu_F$ on $(\mathbb{R}^2, \mathcal{B}^2)$ such that

$$
\mu_F((-\infty, x] \times (-\infty, y]) = F(x, y) \text{ for all } x, y.
$$

6. Let $f_n : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}), n \geq 1$ be a sequence of measurable functions such that $f_n(\omega) \to f(\omega)$ as $n \to \infty$ for every $\omega \in \Omega$. Show that $f : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B})$ is also measurable, i.e., the class of measurable functions is closed under point-wise limit.

7. Show that if $F(x) = \mathbb{P}(X \leq x)$ is continuous for all $x$ then $Y = F(X)$ has a uniform distribution on $(0,1)$, i.e., $\mathbb{P}(Y \leq y) = y$ for $0 \leq y \leq 1$. 