Homework 1

MATH 561/STAT 551: Theory of Probability I

Due date: January 24, 2018

Each problem is worth 10 points and only five randomly chosen problems will be graded. Please indicate whom you worked with, it will not affect your grade in any way.

1. Let \( \mathcal{F}_n \) be classes of subsets of \( S \). Suppose each \( \mathcal{F}_n \) is a field, and \( \mathcal{F}_n \subseteq \mathcal{F}_{n+1} \) for \( n = 1, 2, \ldots \). Define \( \mathcal{F} = \bigcup_{i=1}^{\infty} \mathcal{F}_n \). Show that \( \mathcal{F} \) is a field. Give an example to show that, if each \( \mathcal{F}_n \) is a \( \sigma \)-field, then \( \mathcal{F} \) need not be a \( \sigma \)-field.

**Extra:** Show that \( \bigcup_{i=1}^{\infty} \mathcal{F}_n \) is never a \( \sigma \)-field (when \( \mathcal{F}_n \)'s are strictly increasing).

**Solution:** (a) It is obvious that \( \emptyset \in \mathcal{F} \). Take \( A \in \mathcal{F} \), then \( \exists n \in \mathbb{N} \) s.t. \( A \in \mathcal{F}_n \). This implies \( A^c \in \mathcal{F}_n \subseteq \mathcal{F} \). Now take \( B \in \mathcal{F} \), then \( \exists m \in \mathbb{N} \) s.t. \( B \in \mathcal{F}_m \). We have \( A \cup B \in \mathcal{F}_{m \lor n} \) since \( (\mathcal{F}_n)_{n \in \mathbb{N}} \) is increasing. Therefore, \( \mathcal{F} \) is a field.

(b) Consider \( \Omega = \mathbb{N} \) and \( \mathcal{F}_n = \sigma(\{k\}; \, k \leq n) \) (i.e. all the subsets of \( \{1, \ldots, n\} \) and their complements) : \( (\mathcal{F}_n)_{n \in \mathbb{N}} \) is increasing. Observe that \( \{2n\} \subseteq \mathcal{F}_{2n} \subset \mathcal{F} \) but \( 2\mathbb{N} \notin \mathcal{F} \). If \( 2\mathbb{N} \notin \mathcal{F} \), then \( 2\mathbb{N} \in \mathcal{F}_n \) for some \( n \geq 1 \). But \( \mathcal{F}_n = \{A, A \cup \{n+1, n+2, \ldots\} \mid A \subseteq \{1, 2, \ldots, n\}\} \) and \( \not\in 2\mathbb{N} \).

**EXTRA:** The proof generalizes the example given above and is due to Huff Broughton.

---

First we claim that, there exists an infinite subsequence \( n_1 < n_2 < n_3 < \cdots \) and a sequence of disjoint sets \( B_2, B_3, \ldots \) such that \( B_k \in \mathcal{F}_{n_k} \setminus \mathcal{F}_{n_{k-1}}, k \geq 2 \).

The proof is via induction and subsequence argument.

Choose \( l \) such that \( \mathcal{F}_1 \neq \emptyset, \Omega \). Take \( A \in \mathcal{F}_1 \setminus \emptyset, \Omega \). It is easy to see that, for every \( k \geq l \) either \( \{A \cap C \mid C \in \mathcal{F}_k\} \neq \{A \cap C \mid C \in \mathcal{F}_{k+1}\} \) or \( \{A^c \cap C \mid C \in \mathcal{F}_k\} \neq \{A^c \cap C \mid C \in \mathcal{F}_{k+1}\} \). If not, assume that for some \( k \geq l \) we have \( \{A \cap C \mid C \in \mathcal{F}_k\} = \{A \cap C \mid C \in \mathcal{F}_{k+1}\} \) and \( \{A^c \cap C \mid C \in \mathcal{F}_k\} = \{A^c \cap C \mid C \in \mathcal{F}_{k+1}\} \). Then for \( C \in \mathcal{F}_{k+1} \setminus \mathcal{F}_k \) we have \( A \cap C \in \mathcal{F}_k \subseteq \mathcal{F}_{k+1} \subseteq \mathcal{F}_{k+1} \) and \( A^c \cap C \in \mathcal{F}_k \). Thus \( \mathcal{F} \) is a field.

Now use the same argument on the strictly increasing \( \sigma \)-fields \( \{A \cap C \mid C \in \mathcal{F}_{n_k}\}, k \geq 1 \) to get \( A_2 \subseteq A_1 \) and a further increasing subsequence \( n_{k2}, k \geq 1 \) such that \( \{A_2 \cap C \mid C \in \mathcal{F}_{n_{k2}}\} \subseteq \{A_2 \cap C \mid C \in \mathcal{F}_{n_{k2+1}}\} \) for all \( k \geq 1 \).

Inductively, we can get (check the details) using the subsequence argument a subsequence \( n_{k} \), and a sequence of disjoint subsets such that \( A_k \in \mathcal{F}_{n_k} \setminus \mathcal{F}_{n_{k-1}}, k \geq 2 \). Finally take \( B_k = A_{k-1} \setminus A_k, k \geq 2 \).

We define the function \( \pi : \Omega \rightarrow \mathbb{N} \) by

\[
\pi(x) := \sum_{k=1}^{\infty} k \cdot \mathbb{I}_{B_k}(x).
\]

Note that \( \pi \) is well defined and finite valued. Define the collection of subsets

\[
\mathcal{G}_n := \{A \subseteq \mathbb{N} \mid \pi^{-1}(A) \in \mathcal{F}_{n-1}, n \in A\}.
\]

We claim that, for every \( n \geq 1 \) there exists a smallest set \( T_n \in \mathcal{G}_n \). Note that \( \mathbb{N} \in \mathcal{G}_n \) as \( \pi^{-1}(\mathbb{N}) = \Omega \).
and \( n \in \mathbb{N} \). So, take \( T_n \) as the intersection of all sets in \( G_n \). First of all, \( T_n \) can be written as intersection of countable subsets from \( G_n \) and thus is in \( G_n \). Note that \( T_n \neq \{n\} \) as \( \pi^{-1}(\{n\}) = B_n \in \mathcal{F}_n \setminus \mathcal{F}_{n-1} \) and \( T_n \subseteq \{n, n+1, n+2, \ldots \} \) as \( \pi^{-1}(\{n, n+1, n+2, \ldots \}) = \cup_{k \geq n} B_k = A_{n-1} \in \mathcal{F}_{n-1} \). Also, if \( m \in T_n \) then \( T_m \subseteq T_n \), as \( T_m \cap T_n \supseteq T_m \), by minimality and \( m \in T_n \setminus T_m \).

Take \( m_1 = 1 \) and inductively define a subsequence \( m_2 < m_3 < \ldots \) such that \( m_k+1 \in T_{n_k} \). This is possible since \( \emptyset \neq T_n \setminus \{n\} \subseteq \{n+1, n+2, \ldots \} \). Clearly,

\[
T_{m_1} \supseteq T_{m_2} \supseteq T_{m_3} \supseteq \cdots
\]

Finally take

\[
E = \bigcup_{k \geq 1} B_{m_{2k}}.
\]

We claim that, \( E \notin \mathcal{F}_n \) for all \( n \geq 1 \). Thus \( \mathcal{F}_n \setminus \mathcal{F}_n \) is not a \( \sigma \)-algebra. Suppose there exists \( n \) such that \( E \in \mathcal{F}_n \). We can find \( m_{2k} \geq n \) such that \( E \in \mathcal{F}_{m_{2k}} \) (the \( \sigma \)-fields are increasing). Now

\[
\pi^{-1}(\{m_{2j} | j \geq k\}) = \bigcup_{j \geq k} B_{m_{2j}} = E \setminus \bigcup_{j=1}^{k-1} B_{m_{2j}} \in \mathcal{F}_{m_{2k}}.
\]

Thus \( m_{2k} \in \{m_{2j} | j \geq k\} \in G_{m_{2k}} \) and \( T_{m_{2k}} \subseteq \{m_{2j} | j \geq k\} \). This contradicts our choice that \( m_{2k+1} \in T_{m_{2k}} \).

2. Given a non-empty collection \( C \) of sets, we defined \( \mathcal{A}(C) \) as the intersection of all fields containing \( C \). Show that \( \mathcal{A}(C) \) is the class of sets of the form \( \bigcap_{i=1}^{n} \bigcap_{j=1}^{n} A_{i,j} \) where for each \( i \) and \( j \) either \( A_{i,j} \in C \) or \( A_{i,j}^c \in C \), and where the \( m \) sets \( \bigcap_{j=1}^{n} A_{i,j}, i = 1, 2, \ldots, m \) are disjoint.

**Solution:** Denote \( \mathcal{G} \) the class of the sets mentioned in the problem. We would like to show that \( \mathcal{A}(C) = \mathcal{G} \). It’s straightforward to see that \( \mathcal{A}(C) \) contains \( \mathcal{G} \) (closed under complement, finite intersection and union). Thus \( \mathcal{G} \subseteq \mathcal{A}(C) \). It is enough to prove that \( \mathcal{G} \) itself is a field. Observe that \( (\bigcup_{j=1}^{m_{1}} \bigcap_{i=1}^{m_{1}} A_{i,j}) \cap (\bigcup_{k=1}^{m_{2}} \bigcap_{l=1}^{m_{2}} B_{k,l}) = \bigcup_{j \in m_{1}, k \in m_{2}} (\bigcap_{i=1}^{m_{1}} A_{i,j} \cap B_{k,l}) \) is closed under finite intersection. In addition, \( (\bigcup_{j=1}^{m_{1}} \bigcap_{i=1}^{m_{1}} A_{i,j})^c = \bigcap_{j=1}^{m_{1}} \bigcup_{i=1}^{m_{1}} A_{i,j}^c \) and \( \bigcap_{j=1}^{m_{1}} A_{i,j}^c \) is closed under finite intersection. We have thus proved the desired result.

3. Suppose \( B \in \sigma(C) \), for some collection \( C \) of subsets. Show there exists a countable subcollection \( C_B \) of \( C \) such that \( B \in \sigma(C) \).

**Solution:** Denote \( \mathcal{F} = \bigcup \sigma(A) \) where \( A \) runs over the set of all countable subsets of \( C \). The question consists in proving that \( \mathcal{F} = \sigma(C) \). It is obvious that \( \mathcal{F} \subseteq \sigma(C) \). Thus it suffices to show that \( \mathcal{F} \) is a \( \sigma \)-field. Take \( A \in \mathcal{F} \). There exists countable subset \( A \) s.t. \( A \in \sigma(A) \). Then \( A^c \in \sigma(A) \subseteq \mathcal{F} \). Now take \( (A_n)_{n \in \mathbb{N}} \in \mathcal{F}^\mathbb{N} \) : \( \exists A_n \in \sigma(A_n) \). We have then \( \bigcup_{i=1}^{\infty} A_n \in \sigma(\bigcup_{i=1}^{\infty} A_n) \subseteq \mathcal{F} \). Thus \( \mathcal{F} \) is a \( \sigma \)-field.

4. Show that, in the definition of “a probability measure \( \mu \) on a measurable space \((\Omega, \mathcal{F})\)”, we may replace “countably additive” by “finitely additive, and satisfies

\[
\text{“if } A_n \downarrow \emptyset \text{ then } \mu(A_n) \to 0. \text{”}
\]

**Solution:** It is straightforward that countably additive implies finite additivity and continuity from above (in particular, if \( A_n \downarrow \emptyset \) then \( \mu(A_n) \to 0 \)). Conversely, consider \( (A_n)_{n \in \mathbb{N}} \) disjoint. We have \( \mu(\bigcup_{n \in \mathbb{N}} A_n) = \mu(\bigcup_{n \in \mathbb{N}} A_n \setminus \bigcup_{n \in \mathbb{N}} A_n) + \sum_{n \in \mathbb{N}} \mu(A_n) \) by finite additivity. Note that the first term goes to 0 and the second converges to \( \sum_{n \in \mathbb{N}} \mu(A_n) \) by hypotheses. Thus we obtain countable additivity.

5. Let \( B \) be the Borel subsets of \( \mathbb{R} \). For \( B \in B \) define

\[
\mu(B) = \begin{cases} 
1 & \text{if } (0, \varepsilon) \subset B \text{ for some } \varepsilon > 0 \\
0 & \text{otherwise.}
\end{cases}
\]

(a) Show that \( \mu \) is not finitely additive on \( B \).

(b) Show that \( \mu \) is finitely additive but not countably additive on the field \( B_0 \) of finite disjoint unions of intervals \((a, b)\).
Solution: (a) Note that \( \mu(Q \cap (0,1)) = \mu(Q^c \cap (0,1)) = 0 \), but \( \mu((0,1)) = 1 \). Thus \( \mu \) is not finitely additive on \( B \). (b) Consider \( B = \bigcup_{i=1}^{n}(a_i, b_i) \) and \( B' = \bigcup_{i=1}^{m}(a'_i, b'_i) \) disjoint. We can check that at most one of them has \( \mu \) measure equal to 1 (otherwise, \( \exists \varepsilon > 0 \) s.t. \((0, \varepsilon) \in B \cap B'\), which contradicts disjointness). This leads to \( \mu(B) + \mu(B') = \mu(B \cup B') \): \( \mu \) is finitely additive on \( B_0 \). In addition, for \( k \in \mathbb{N} \), \( \mu((\frac{1}{k+1}, \frac{1}{k}]) = 0 \), but \( \mu(\bigcup_{k=1}^{\infty}(\frac{1}{k+1}, \frac{1}{k}]) = 1 \): \( \mu \) is not countable additive on \( B_0 \).

6. Give an example of a measurable space \((\Omega, \mathcal{F})\), a collection \( \mathcal{A} \) and probability measures \( \mu \) and \( \nu \) such that
   i) \( \mu(A) = \nu(A) \) for all \( A \in \mathcal{A} \).
   ii) \( \mathcal{F} = \sigma(\mathcal{A}) \).
   iii) \( \mu \neq \nu \).

   Hint: Start with \( \Omega = \{1, 2, 3, 4\} \).

   Solution: Take \( \Omega = \{1, 2, 3, 4\} \) and \( \mathcal{F} = \mathcal{P}(\Omega) \). Consider \( \mathcal{A} = \{\{1, 2\}, \{1, 3\}\} \), we have \( \mathcal{F} = \sigma(\mathcal{A}) \). Set \( \mu(\{1\}) = \mu(\{4\}) = \frac{1}{6}, \mu(\{2\}) = \mu(\{3\}) = \frac{1}{3} \) and \( \nu(\{1\}) = \nu(\{2\}) = \nu(\{3\}) = \nu(\{4\}) = \frac{1}{4} \) s.t. \( \mu \neq \nu \) but \( \mu = \nu \) on \( \mathcal{A} \).

7. Let \( \mu \) be a probability measure on \((\Omega, \mathcal{F})\), where \( \mathcal{F} = \sigma(\mathcal{A}) \) for a field \( \mathcal{A} \). Show that for each \( B \in \mathcal{F} \) and \( \varepsilon > 0 \) there exists \( A \in \mathcal{A} \) such that \( \mu(B \Delta A) < \varepsilon \). Here \( B \Delta A = (B \setminus A) \cup (A \setminus B) \) is the symmetric difference between \( A, B \).

   Hint: Consider the collection of all fields satisfying this property.

   Solution: Denote \( \mathcal{T} = \{B \in \mathcal{F} \text{ s.t. } \forall \varepsilon > 0, \exists A \in \mathcal{A} \text{ s.t. } \mu(B \Delta A) < \varepsilon\} \). Clearly, \( \emptyset \in \mathcal{T} \). Moreover, \( \mathcal{T} \) is closed under complement since \( B^c \Delta A^c = B \Delta A \). Now consider, \( B = \bigcup_{n=1}^{\infty} B_n \) where \( B_n \in \mathcal{T} \) for all \( n \). Given \( \varepsilon > 0 \), take \( N \in \mathbb{N} \) s.t. \( \mu(B \setminus \bigcup_{n=1}^{N} B_n) \leq \varepsilon/2 \). For \( n \leq N \), take \( A_n \) s.t. \( \mu(B_n \Delta A_n) < \varepsilon/2N \). Since, \( \bigcup_{n=1}^{N} B_n \Delta \bigcup_{n=1}^{N} A_n \subseteq \bigcup_{n=1}^{N} A_n \Delta B_n \), we have \( \mu(B \Delta \bigcup_{n=1}^{N} A_n) < \varepsilon \). Thus \( \mathcal{T} \) is a \( \sigma \)-field containing \( \mathcal{A} \), which concludes the proof.