Homework 12

Math 561: Theory of Probability I

Due date: April 25, 2018

Each problem is worth 10 points and only five randomly chosen problems will be graded if there are more than 5 problems. Please indicate whom you worked with, it will not affect your grade in any way.

1. Let \((X_n, \mathcal{F}_n)_{n \geq 0}\) be a submartingale and \(N\) be a stopping time w.r.t. \((\mathcal{F}_n)_{n \geq 0}\) such that \(N \leq \ell\) a.s.
   
   (i) Prove that for any \(j \leq \ell\), we have \(X_j \mathbf{1}_{N=j} \leq \mathbb{E}(X_{j} \mathbf{1}_{N=j} \mid \mathcal{F}_j)\) and use this to conclude that \(X_N \leq \mathbb{E}(X_{\ell} \mid \mathcal{F}_N)\) and \(\mathbb{E}(X_N) \leq \mathbb{E}(X_{\ell})\).
   
   (ii) Furthermore, if \(T\) is another stopping time w.r.t. \((\mathcal{F}_n)_{n \geq 0}\) such that \(T \leq N\) a.s., prove that \(\mathbb{E}(X_T) \leq \mathbb{E}(X_N)\).

2. Let \(S_n = \sum_{i=1}^{n} \xi_i\), where the \((\xi_i)\) are independent, \(\mathbb{E}(\xi_i) = 0\) and \(\text{Var}(\xi_i) < \infty\). Let \(s_n^2 = \sum_{i=1}^{n} \text{Var}(\xi_i)\). So we know that \((S_n^2 - s_n^2)\) is a martingale. Suppose also that \(|\xi_i| \leq K\) for some constant \(K\). Show that

\[
\mathbb{P}(\max_{m \leq n} |S_m| \leq x) \leq s_n^{-2}(K + x)^2, \quad x > 0.
\]

**Hint:** Work with the second moment martingale and appropriate stopping time.

3. Let \((X_n)\) be a martingale with \(X_0 = 0\) and \(\mathbb{E}X_n^2 < \infty\). Using the fact that \((X_n + c)^2\) is a sub-martingale, show that

\[
\mathbb{P}(\max_{m \leq n} X_m \geq x) \leq \frac{\mathbb{E}(X_n^2)}{x^2 + \mathbb{E}(X_n^2)}, \quad x > 0.
\]

4. Let \(S_n, n \geq 0\) be the symmetric random walk (SRW) on \(\mathbb{Z}\) with \(S_0 = 1, S_n = S_{n-1} + \xi_n\) where \(\xi_1, \xi_2, \ldots\) are i.i.d. with \(\mathbb{P}(\xi_1 = +1) = 1/2 = \mathbb{P}(\xi_1 = -1)\). Let \(N = \inf\{n \geq 0 \mid S_n = 0\}\) be the hitting time of 0.

   (i) Prove that \(\mathbb{P}(\sup_{n \geq 1} S_{N \wedge n} \geq m) = \frac{1}{m}\) for all \(m > 0\).

   (ii) Use (i) to show that \(N < \infty\) a.s. This proves that \(S_{N \wedge n} \to 0\) a.s.

**Note:** This implies that for a SRW starting at 1, it will come back to 0 a.s. and in between the path will go upto height \(H\) having distribution \(\mathbb{P}(H \geq m) = 1/m, m \geq 1\). In fact the whole piecewise linear path, after appropriate scaling, will converge to a continuous path, called Brownian excursion (to be proved in 562).

5. Suppose \(\mathcal{F}_n \uparrow \mathcal{F}_\infty\) and \(Y_n \to Y_\infty\) in \(L^1\). Show that \(\mathbb{E}(Y_n \mid \mathcal{F}_n) \to \mathbb{E}(Y_\infty \mid \mathcal{F}_\infty)\) in \(L^1\).

6. Let \((X_n, \mathcal{F}_n)_{n \geq 0}\) be a positive sub-martingale with \(X_0 = 0\). Let \((V_n)_{n \geq 1}\) be a predictable sequence of random variables such that

\[
B \geq V_1 \geq V_2 \geq \cdots \geq 0, \text{ for some constant } B.
\]

Prove that for \(\lambda > 0\),

\[
\mathbb{P}(\max_{1 \leq j \leq n} V_j X_j \geq \lambda) \leq \lambda^{-1} \sum_{j=1}^{n} \mathbb{E}(V_j(X_j - X_{j-1})).
\]
7. Let \((X_n, \mathcal{F}_n)_{n \geq 0}\) be martingale and write \(\Delta_n = X_n - X_{n-1}\). Suppose that \(b_m \uparrow \infty\) and \(\sum_{m=1}^{\infty} b^{-2}_m \mathbb{E} \Delta^2_m < \infty\). Prove that \(X_n/b_n \rightarrow 0\) a.s.