Joint distribution of Maximum and Minimum

Let $X_1, X_2, \ldots, X_n$ are independent random variables each having CDF $F(\cdot)$. What is the joint pdf of the maximum $U = \max\{X_1, X_2, \ldots, X_n\}$ and the minimum $V = \min\{X_1, X_2, \ldots, X_n\}$?
Expectation of a function of rvs

**Useful fact**

Let $g(x, y)$ be a function. If $X, Y$ are discrete with p.m.f. $p(x, y)$, then

$$
\mathbb{E} g(X, Y) = \sum_x \sum_y g(x, y)p(x, y).
$$

If $X, Y$ are jointly continuous with density $f(x, y)$, then

$$
\mathbb{E} g(X, Y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y)f(x, y)dx dy.
$$
Example

Let $X$ and $Y$ be independent variables uniformly distributed on $(0, 1)$. Find $\mathbb{E}|X - Y|^\alpha$.

Joint density: $f(x, y) = 1$ if $0 < x < 1, 0 < y < 1$ and $0$ otherwise.

Consequences

Consequence 1

$\mathbb{E}(X + Y) = \mathbb{E}X + \mathbb{E}Y$ and more generally $\mathbb{E} \sum_{i=1}^n X_i = \sum_{i=1}^n \mathbb{E}X_i$.

Consequence 2

If $X$ and $Y$ are independent and $g(x), h(y)$ are two functions, then

$\mathbb{E}(g(X)h(Y)) = \mathbb{E}g(X) \cdot \mathbb{E}h(Y)$. 
Expected number of events that occur

It is a common situation where we want to compute \( \mathbb{E}X \) with \( X \) being the number of something. Moreover, it is often the case that for some events \( A_1, A_2, \ldots, A_n \), \( X \) is the number of these events that occur (e.g. \( A_i = \{ \text{success on trial } i \} \)). Then,

\[
X = \sum_{i=1}^{n} I_{A_i} \quad \text{with} \quad I_{A_i} = \begin{cases} 
1, & \text{if } A_i \text{ occurs,} \\
0, & \text{if not}
\end{cases}
\]

and

\[
\mathbb{E}X = \sum_{i=1}^{n} \mathbb{E}I_{A_i} = \sum_{i=1}^{n} \left( 1 \cdot \mathbb{P}(A_i) + 0 \cdot \mathbb{P}(A_i^c) \right) = \sum_{i=1}^{n} \mathbb{P}(A_i).
\]

Example 2j

Ten hunters are waiting for ducks to fly by. When a flock of ducks flies overhead, the hunters are at the same time, but each chooses his target at random, independently of the others. If each hunter independently hits his target with probability \( p \), compute the expected number of ducks that escape unhurt when a flock of size \( f \) flies overhead.
Second-order moments of number of events that occur

For some events $A_1, A_2, \ldots, A_n$, let

$$X = \sum_{i=1}^{n} I_{A_i} \quad \text{with} \quad I_{A_i} = \begin{cases} 1, & \text{if } A_i \text{ occurs,} \\ 0, & \text{if not} \end{cases}$$

be the number of these events that occur. Note that

$$\frac{X(X - 1)}{2} = \binom{X}{2} = \sum_{i_1 < i_2} I_{A_{i_1}} I_{A_{i_2}}$$

is the number of pairs of events $A_1, A_2, \ldots, A_n$ where both events occur.

Binomial Distribution

Let $X$ be a binomial random variable with parameters $n$ and $p$. Compute $\mathbb{E}X$, $\mathbb{E}X(X - 1)$ and $\text{Var}(X)$.
**Variances, Covariances**

### Covariance between $X$ and $Y$

$$\text{Cov}(X, Y) = \mathbb{E}(X - \mathbb{E}X)(Y - \mathbb{E}Y) = \mathbb{E}XY - \mathbb{E}X \cdot \mathbb{E}Y$$

**Idea:** Covariance gives idea of the relationship between $X$ and $Y$.

**Some basic properties:**

$$\text{Cov}(X, Y) = \text{Cov}(Y, X), \quad \text{Cov}(X, X) = \text{Var}(X), \quad \text{Cov}(aX + b, Y) = a \text{Cov}(X, Y).$$

### Important property

$$\text{Cov}\left(\sum_{i=1}^{n} X_i, \sum_{j=1}^{n} Y_j\right) = \sum_{i=1}^{n} \sum_{j=1}^{n} \text{Cov}(X_i, Y_j)$$

**Consequence:** $\text{Var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \text{Var}(X_i) + 2 \sum_{i<j} \text{Cov}(X_i, X_j)$

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**Correlation Coefficient**

### Correlation between $X$ and $Y$

$$\rho(X, Y) = \text{Corr}(X, Y) = \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X) \text{Var}(Y)}}$$

**Fact 1:** $-1 \leq \rho(X, Y) \leq 1$

**Terminology:** $X$ and $Y$ are called **uncorrelated** when $\rho(X, Y) = 0$.

**Fact 2:** For example, if $\rho(X, Y) = \pm 1$, then $Y = \pm aX + b$ with $a > 0$.

**Fact 3:** $\rho(aX + b, Y) = \rho(X, Y)$ $(a > 0)$

**Note:** $\rho(X, Y)$ measures the strength and direction of a linear relationship between $X$ and $Y$ (close to 1: strong linear, positive slope; close to $-1$: strong linear, negative slope).
**Example**

Toss a fair coin 3 times. Let $X$ be the number of heads, and $Y$ be the number of tails. Find $\text{Cov}(X, Y)$.

**Problem 39**

Let $X_1, X_2, \ldots$ be independent random variables with common mean $\mu$ and common variance $\sigma^2$. Set $Y_n = X_n + 2X_{n+1}$, $n \geq 1$. For $j \geq 0$, find $\text{Cov}(Y_n, Y_{n+j})$ and $\text{Corr}(Y_n, Y_{n+j})$. 
Properties of expectation

**Note 1:** If $X_i$'s are pairwise independent, then

$$\text{Var} \left( \sum_{i=1}^{n} X_i \right) = \sum_{i=1}^{n} \text{Var}(X_i).$$

**Note 2:** If $X_i = 1$ if event $A_i$ occurs and $= 0$ otherwise, then

$$\text{Var} \left( \sum_{i=1}^{n} X_i \right) = 2 \sum_{i<j} \text{P}(A_i \cap A_j) + \sum_i \text{P}(A_i) - \left( \sum_i \text{P}(A_i) \right)^2.$$

**Problem 42**

A group of 20 people consisting of 10 men and 10 women is randomly arranged into 10 pairs of 2 each. Compute the expectation and variance of the number of pairs that consist of a man and a woman.
Example 2l (A random walk in the plane)

Consider a particle initially located at a given point in the plane, and suppose that it undergoes a sequence of steps of fixed length, but in a completely random direction.

Specifically, suppose that the new position after each step is one unit of distance from the previous position and at an angle of orientation from the previous position that is uniformly distributed over $(0, 2\pi)$.

Compute the expected square of the distance from the origin after $n$ steps.