Recap

- If $X, Y$ are independent, $X$ has density $f_X$, $Y$ has density $f_Y$, then the density of $X + Y$ is:

  $$f_{X+Y}(a) = (f_X * f_Y)(a) = \int \mathbb{R} f_X(t)f_Y(a-t)dt.$$  

- **Discrete Case.** Conditional pmf and cdf of $X$ given $Y = y$ ($y$ fixed):

  $$p_{X|Y}(x \mid y) = \frac{p(x, y)}{p_Y(y)}, \quad F_{X|Y}(a \mid y) = \sum_{x \leq a} p_{X|Y}(x \mid y)$$

  for all $y$ with $p_Y(y) > 0$.

- **Continuous case.** Conditional density and cdf of $X$ given $Y = y$ ($y$ fixed):

  $$f_{X|Y}(x \mid y) = \frac{f(x, y)}{f_Y(y)}, \quad F_{X|Y}(a \mid y) = \int_{-\infty}^{a} f_{X|Y}(x \mid y)dx$$

  for all $y$ with $f_Y(y) > 0$. 
Bivariate Normal distribution

Jointly continuous random variables $X$ and $Y$ are bivariate normal if their density is

$$f(x, y) = \frac{1}{2\pi \sigma_X \sigma_Y \sqrt{1 - \rho^2}} e^{-\frac{1}{2(1-\rho^2)} \left( \frac{(x-\mu_X)^2 + (y-\mu_Y)^2}{\sigma_X \sigma_Y} - 2\rho \frac{(x-\mu_X)(y-\mu_Y)}{\sigma_X \sigma_Y} \right)},$$

for $x, y \in \mathbb{R}$, where $\sigma_X, \sigma_Y > 0$, $\rho \in (-1, 1)$, $\mu_X, \mu_Y \in \mathbb{R}$.

Find $f_{X|Y}(x|y)$.

Notation:

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \mathcal{N} \left( \begin{pmatrix} \mu_X \\ \mu_Y \end{pmatrix}, \begin{pmatrix} \sigma^2_X & \rho \sigma_X \sigma_Y \\ \rho \sigma_X \sigma_Y & \sigma^2_Y \end{pmatrix} \right).$$

Here, $\mu_X = \mathbb{E}X$, $\mu_Y = \mathbb{E}Y$, $\sigma^2_X = \text{Var}(X)$, $\sigma^2_Y = \text{Var}(Y)$. The parameter $\rho$ accounts for dependence and will be clarified in the next chapter.
Joint distribution of Maximum and Minimum

Let \( X_1, X_2, \ldots, X_n \) are independent random variables each having CDF \( F(\cdot) \). What is the joint pdf of the maximum \( U = \max\{X_1, X_2, \ldots, X_n\} \) and the minimum \( V = \min\{X_1, X_2, \ldots, X_n\} \)?

Distribution of function of a random variable

**Basic problem:** Suppose \( X \) is continuous with density \( f(x) \). Consider \( Y = g(X) \) for some invertible function \( g \), for example, \( g(x) = e^x \). What is the density of \( Y \)?
Joint Distribution of functions of random variables

**Basic problem:** Suppose $X_1, X_2$ are jointly continuous with density $f(x, y)$. Consider $Y_1 = g_1(X_1, X_2), Y_2 = g_2(X_1, X_2)$.

E.g. $g_1(x_1, x_2) = x_1 + x_2$ and $g_2(x_1, x_2) = x_1 - x_2$.

What is the density of $Y_1, Y_2$?

**One variable versus two variables:**

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**Basic problem**

Suppose $X_1, X_2$ are jointly continuous with density $f(x, y)$. Consider $Y_1 = g_1(X_1, X_2), Y_2 = g_2(X_1, X_2)$. Find the density of $Y_1, Y_2$. 
Example 7a

Let $X_1$ and $X_2$ be jointly continuous random variables with probability density function $f_{X_1,X_2}$. Let $Y_1 = X_1 + X_2$, $Y_2 = X_1 - X_2$. Find the joint density function of $Y_1$ and $Y_2$ in terms of $f_{X_1,X_2}$. 
Example 7b

Let \((X, Y)\) denote a random point in the plane, and assume that the rectangular coordinates \(X\) and \(Y\) are independent standard normal random variables.

What is the joint distribution of \(R, \Theta\), the polar coordinate representation of \((X, Y)\).