Math 461: Introduction to Probability
Lecture 10: Random Variables II

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Recap

- A Random Variable (rv) $X$ is a real-valued function on the sample space.
- Discrete rvs take finite or countably infinite many values.
- Characterize a rv by specifying the possible values and probability for each value (pmf).
- Expectation or Mean of a rv $X$ is defined as
  \[ \mathbb{E}X = \sum_i x_i p(x_i). \]
- Variance of a rv $X$ is defined as
  \[ \text{Var}(X) = \mathbb{E}(X - \mathbb{E}X)^2. \]
Computing Expectation

Computing expectation of function of random variable

If $X$ is a discrete random variable taking values $x_i$ with probability $p(x_i)$, and $g$ is a function, then

$$
\mathbb{E}g(X) = \sum_i g(x_i)p(x_i).
$$

Terminology: $\mathbb{E}X = \text{mean, 1st moment}$; $\mathbb{E}X^n = \text{n-th moment, } n \geq 1$.

Simple consequence: for two constants $a, b$,

$$
\mathbb{E}(aX + b) = \sum_i (ax_i + b)p(x_i) = a\sum_i x_ip(x_i) + b\sum_i p(x_i) = a\mathbb{E}X + b.
$$

Example 4b

- A product that is sold seasonally yields a net profit of $b$ dollars for each unit sold and a net loss of $\ell$ dollars for each unit left unsold when the season ends.
- The number of units of the product that are ordered at a specific department store during any season is a random variable having probability mass function $p(i)$, $i \geq 0$.
- If the store must stock this product in advance, determine the number of units the store should stock so as to maximize its expected profit.

Let $P_s$ be the total profit if when the stock is $s$. Let $X$ be the demand or the number of units ordered with pmf $p(i)$, $i = 0, 1, \ldots$. Then

$$
P_s = \begin{cases} 
  bX - \ell(s - X) = bs + (b + \ell)(X - s) & \text{if } X \leq s, \\
  bs & \text{if } X > s.
\end{cases}
$$

Choose $s$ for which $\mathbb{E}(P_s)$ is maximum.
Some Important Facts

**Note 1:** \( \text{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}X)^2 \).  

**Example:** \( X \) is the outcome when a fair die is rolled.

**Note 2:** for two numbers \( a, b \), \( \text{Var}(aX + b) = a^2 \text{Var}(X) \).
Bernoulli Random Variables

**Bernoulli experiment:**
2 possible outcomes:
- success (with probability $p$) and
- failure (with probability $1 - p$)

**Bernoulli random variable**
$X$ takes 2 values
- 1 (success) with probability $p$ and
- 0 (failure) with probability $1 - p$.

**Example:** tossing a fair coin; $H = $ success, $T = $ failure; $p = 1/2$.

Binomial Random Variables

**Binomial random variable $X$**
Number of successes in $n$ independent Bernoulli trials:

$$P(X = i) = \binom{n}{i} p^i (1-p)^{n-i}, \quad i = 0, 1, \ldots, n.$$ 

**Notation**
$X \sim \text{Bin}(n, p)$.
Bernoulli: $X \sim \text{Bin}(1, p)$.

**Example:** tossing a fair coin $n$ times; $X = $ number of $H$’s $= \text{Bin}(n, 1/2)$. 
Questions?

Properties of Binomial RVs

If $X \sim \text{Bin}(n, p)$, what about the shape of its probability mass function?
Mean of Binomial RVs

If $X \sim \text{Bin}(n, p)$, then $\mathbb{E}X = np$.

Variance of Binomial RVs

Similarly, $\mathbb{E}X(X - 1) = n(n - 1)p^2$ and hence

$$\text{Var}(X) = np(1 - p).$$
Problem 4.42

- In flight, airplane engines will fail with probability $1 - p$, independently from engine to engine.
- An airplane needs a majority of its engines operative to complete a successful flight.

For what values of $p$ is a 5-engine plane preferable to a 3-engine plane?

Solution
Problem 4.48

- It is known that diskettes produced by a certain company will be defective with probability .01, independently of each other.
- The company sells the diskettes in packages of size 10 and offers a money-back guarantee that at most 1 of the 10 diskettes in the package will be defective.
- The guarantee is that the customer can return the entire package of diskettes if he or she finds more than one defective diskette in it.

If someone buys 3 packages, what is the probability that he or she will return exactly 1 of them?

Let $p$ be the probability that a single package contains more than one defective diskette. Then

\[ p = \]

Solution
Problem 4.41

- A man claims to have extrasensory perception (ESP).
- As a test, a fair coin is flipped 10 times and the man is asked to predict the outcome in advance.
- He gets 7 out of 10 correct.

What is the probability that he would have done at least this well if he had no ESP?

Solution
Questions?

That's all Folks!