HOMEWORK 7
Math 461: Probability Theory

DUE DATE: March 23, 2023

Each problem is worth 10 points and only five randomly chosen problems will be graded. Please indicate whom you worked with, it will not affect your grade in any way.

1. Suppose that an unbiased coin is tossed \(n\) times. Let \(X\) denote the number of times the pattern \(HTH\) appears, for example in the sequence \(HHTHTHT\) the pattern \(HTH\) appears 2 times. Find the expected value of \(X\).

2. Let \(\sigma\) be a random ordering of the sequence \(1, 2, \ldots, n\) (for example \(2, 4, 5, 1, 3, \ldots\)) where each of the \(n!\) ordering has equal probability. Given an ordering we say that the \(i\)-th position is a local maxima if the value at the \(i\)-th position is bigger than the neighboring value/values. For example, if the ordering is \(3, 2, 4, 1, 5\) the 1st, 3rd and 5th positions are local maxima and there are 3 local maxima. Find the expected total number of local maxima in \(\sigma\).

3. Let \(X\) be a random variable with probability density function

\[
f(x) = \begin{cases} 
  c(1 - x^2) & -1 < x < 1 \\
  0 & \text{otherwise.}
\end{cases}
\]

(a) What is the value of \(c\)?
(b) What is the cumulative distribution function of \(X\)?

4. A system consisting of one original unit plus a spare can function for a random amount of time \(X\). If the density of \(X\) is given (in units of months) by

\[
f(x) = \begin{cases} 
  C \cdot xe^{-x/2} & x \geq 0 \\
  0 & x \leq 0.
\end{cases}
\]

What is the probability that the system functions for at least 5 months?

5. The probability density function of \(X\), the lifetime of a certain type of electronic device (measured in hours), is given by

\[
f(x) = \begin{cases} 
  \frac{10}{x^2} & x > 10 \\
  0 & x \leq 10.
\end{cases}
\]

(a) Find \(P(X > 20)\).
(b) What is the cumulative distribution function of \(X\)?
(c) What is the probability that, of 6 such types of devices, at least 3 will function for at least 15 hours? What assumptions are you making?

6. A filling station is supplied with gasoline once a week. If its weekly volume of sales in thousands of gallons is a random variable with probability density function

\[
f(x) = \begin{cases} 
  5(1 - x)^4 & 0 < x < 1 \\
  0 & \text{otherwise.}
\end{cases}
\]

what must the capacity of the tank be so that the probability of the supply’s being exhausted in a given week is .01?

7. Compute \(\mathbb{E}X\) if \(X\) has a density function given by

(a) \(f(x) = \begin{cases} 
  \frac{1}{2}xe^{-x/2} & x > 0 \\
  0 & \text{otherwise.}
\end{cases}\)

(b) \(f(x) = \begin{cases} 
  c(1 - x^2) & -1 < x < 1 \\
  0 & \text{otherwise.}
\end{cases}\)

(c) \(f(x) = \begin{cases} 
  \frac{5}{x^2} & x > 5 \\
  0 & \text{otherwise.}
\end{cases}\)
8. Trains headed for destination A arrive at the train station at 15-minute intervals starting at 7 A.M., whereas trains headed for destination B arrive at 15-minute intervals starting at 7:05 A.M.
(a) If a certain passenger arrives at the station at a time uniformly distributed between 7 and 8 A.M. and then gets on the first train that arrives, what proportion of time does he or she go to destination A?
(b) What if the passenger arrives at a time uniformly distributed between 7:10 and 8:10 A.M.?

9. You arrive at a bus stop at 10 o’clock, knowing that the bus will arrive at some time uniformly distributed between 10 and 10:30.
(a) What is the probability that you will have to wait longer than 10 minutes?
(b) If, at 10:15, the bus has not yet arrived, what is the probability that you will have to wait at least an additional 10 minutes?

10. If X is a normal random variable with parameters $\mu = 10$ and $\sigma^2 = 36$, compute (in terms of the Standard normal CDF $\Phi(\cdot)$)
(a) $P(X > 5)$;  
(b) $P(4 < X < 16)$;  
(c) $P(X < 8)$;  
(d) $P(X < 20)$;  
(e) $P(X > 16)$.