(1) (a) How many vectors \((x_1, x_2, \ldots, x_n)\) are there for which each \(x_i\) is either 0 or 1, and 
\[ x_1 + x_2 + \cdots + x_n = k. \]
(b) How many vectors \((x_1, x_2, \ldots, x_n)\) are there for which each \(x_i\) is either 0 or 1, and 
\[ x_1 + x_2 + \cdots + x_n \leq k. \]
(c) How many vectors \((x_1, x_2, \ldots, x_n)\) are there for which each \(x_i \geq 0\) is a non-negative integer, and 
\[ x_1 + x_2 + \cdots + x_n \leq k. \]

(2) Consider the set \(S\) of numbers \(\{1, 2, \ldots, n\}\). We have proved in class that the number of subsets of \(S\) with size \(k\) is \(\binom{n}{k}\). Count the same number in a different way depending on how many subsets of size \(k\) have \(i\) as their highest numbered member, to give a proof of the following identity known as Fermat’s combinatorial identity: For all integers \(n \geq k\)
\[ \binom{n}{k} = \sum_{i=k}^{n} \binom{i-1}{k-1}. \]

(3) (a) In how many ways can \(n\) identical balls be distributed into \(r\) bins such that each bin contains at least two balls. Assume that \(n \geq 2r\).
(b) Do the same problem as in (a), but now each bin contains at least three balls and \(n \geq 3r\).

(4) Two dice are thrown. Let \(E\) be the event that the sum of the dice is odd, let \(F\) be the event that at least one of the dice lands on 2, and let \(G\) be the event that the sum is 5. Describe the events \(E \cap F\), \(E \cup F\), \(F \cap G\), \(E \cap F^c\), and \(E \cap F \cap G\).

(5) A system is comprised of 5 components, each of which is either working or failed. Consider an experiment that consists of observing the status of each component, and let the outcome of the experiment be given by the vector \((x_1, x_2, x_3, x_4, x_5)\), where \(x_i\) is equal to 1 if component \(i\) is working and is equal to 0 if component \(i\) is failed.
(a) How many outcomes are in the sample space of this experiment?
(b) Suppose that the system will work if components 1 and 2 are both working, or if components 3 and 4 are both working, or if components 1, 3, and 5 are all working. Let \(W\) be the event that the system will work. Specify all the outcomes in \(W\).
(c) Let \(A\) be the event that components 4 and 5 are both failed. How many outcomes are contained in the event \(A\)?
(d) Write out all the outcomes in the event \(A \cap W\).

(6) A group of individuals containing 10 boys and 20 girls is lined up in random order; that is, each of the \(30!\) permutations is assumed to be equally likely. What is the probability that the person in the \(i\)-th position, \(1 \leq i \leq 30\), is a girl?

(7) A die is rolled until either 1 or 6 appears. Find the probability that a 6 occurs first. Simplify the answer. **Hint:** Let \(E_n\) denote the event that a 6 occurs on the \(n\)-th roll and no 1 or 6 occurs on the first \(n-1\) rolls. Find \(P(E_n)\) and express the above probability in terms of them. You can use any other argument too.

(8) A card player is dealt a 13 card hand from a well-shuffled, standard deck of cards. What is the probability that the hand is void in at least one suit (“void in a suit” means having no cards of that suit)? **Hint:** Let \(E_i\) be the event that the hand is void in the suit \(i\) for \(i = 1, 2, 3, 4\) (clubs, hearts, diamonds and spades).

(9) For a group of 10 people, find the probability that all 4 seasons (winter, spring, summer, fall) occur at least once each among their birthdays, assuming that all seasons are equally likely. **Hint:** Let \(E_i\) be the event that there are no birthdays in the \(i\)-th season.