**Def:** Suppose $S$ is a nonempty subset of an inner product space $V$. The orthogonal complement of $S$ is

$$S^\perp = \{ x \in V \mid \langle x, y \rangle = 0 \text{ for all } y \in S \}.$$ 

**Ex:** $(\mathbb{R}^3, \text{dot product})$

- $S_1 = \{ e_1 \}$, $S_1^\perp = \text{span} \{ e_2 \}$
- $S_2 = \text{span} \{ (1,1) \}$, $S_2^\perp = \text{span} \{ (1,-1) \}$

Also, $S \cap S^\perp$ contains at most the zero vector.

**Note:** $S^\perp$ is always a subspace, since if $x_1, x_2 \in S^\perp$ then $\langle c x_1 + x_2, y \rangle = c \langle x_1, y \rangle + \langle x_2, y \rangle = c \cdot 0 + 0 = 0$ for all $y \in S$.

**Thm:** Suppose $W$ is a finite-dim'l subspace of an inner product space $V$. For each $y \in V$, there exist unique vectors $w \in W$ and $z \in W^\perp$ such that

$$y = w + z.$$ 

Moreover, if $\{ u_1, u_2, \ldots, u_k \}$ is an o.n.b. for $W$, then

$$w = \sum_{i=1}^k \langle y, u_i \rangle u_i.$$
This \( w \) is called the orthogonal projection of \( y \) onto \( W \), and gives a linear transformation \( \text{proj}_W : V \rightarrow W \) defined by \( y \mapsto w \).

**Proof of Theorem:** Set \( w = \sum_{i=1}^{k} \langle y, u_i \rangle u_i \) and \( z = y - w \).

Clearly, \( w \in W \) and \( y = w + z \).

Moreover, \( z \in W^\perp \) since for each \( u_j \) we have:

\[
\langle z, u_j \rangle = \langle y - w, u_j \rangle = \langle y, u_j \rangle - \langle w, u_j \rangle = 0.
\]

Thus, we need to prove uniqueness.

Suppose \( w' \in W \) and \( z' \in W^\perp \) with \( y = w' + z' \).

Then, \( w - w' = z' - z \) is in \( W \cap W^\perp = \{0\} \).

So, \( w = w' \) \& \( z = z' \) as needed. \( \quad \text{qed} \).

**Conclusion:** The vector \( w = \text{proj}_W (y) \) above is the closest vector in \( W \) to \( y \) in the following sense:

\[
\| y - x \| \geq \| y - w \| \quad \text{for all} \ x \in W
\]

and equality holds iff \( x = w \).
\[ \| y - x \|^2 = \| \omega + z - x \|^2 \]
\[ = \| (\omega - x) + z \|^2, \quad z \in W^\perp \]
\[ = \| \omega - x \|^2 + 0 + 0 + \| z \|^2 \]
\[ \geq \| z \|^2 = \| y - \omega \|^2 \]
and equality holds iff \( \| \omega - x \|^2 = 0 \) on \( \omega = x \).
\[ \text{qed.} \]

**Regression / least squares fitting.**

**Data:**
\( (x_i, y_i), \quad i = 1, 2, \ldots, n. \)

**Q:** which model \( y = mx + b \) best fits this data?

In \( \mathbb{R}^n \), consider
\[ y = (y_1, \ldots, y_n) \]
\[ x = (x_1, \ldots, x_n) \]
\[ u = (1, \ldots, 1) \]

A perfect fit corresponds to having
\[ y \in \text{span} \{ u, x \}. \]
But in general, \( y \notin \text{span}\{u, x\} = W \)

Natural to define the best fit parameters \((m, b)\) to the scalars with

\[
\text{proj}_W(y) = mx + bu
\]

where the projection is w.r.t. the usual dot product in \(\mathbb{R}^n\). Concretely, this is the same as choosing \(m, b\) to minimize

\[
\sum_{i=1}^{n} (y_i - (mx_i + b))^2
\]

This easily adapts to more complicated models.

**Data:** \((x_i, y_i, z_i), i = 1, \ldots, n\)

**Model:** \(z = ax^2 + bx + cy + d \sin y\)

**Setup:** In \(\mathbb{R}^n\) consider

\[
3 = (3, \ldots, 3_n), \quad u = (x_1^2, \ldots, x_n^2), \quad x = (x_1, \ldots, x_n)
\]

\[
y = (y_1, \ldots, y_n), \quad u = (\sin y_1, \ldots, \sin y_n)
\]

**Best fit** \(\text{proj}_W(3)\) for \(W = \text{span}\{u, x, y, u\}\) is a linear combination \(au + bx + cy + dv\).
How to compute \( \text{proj}_W(z) \) ?

**Thm.** Suppose \( \beta = \{ w_1, \ldots, w_k \} \) is a basis for a subspace \( W \) of \( \mathbb{R}^n \). Let \( A \in M_{k \times n}(\mathbb{R}) \) be the matrix whose rows are \( w_1, \ldots, w_k \).

Then, \[
\left[ \text{proj}_W \right]_\beta^{\text{std. basis}} = (A^t A)^{-1} A^t
\]

where \( \text{proj}_W : \mathbb{R}^n \to W \) is orthogonal projection onto \( W \) w.r.t. the dot product on \( \mathbb{R}^n \).

**pf.** Later