Last time: \( V \) an inner product space.
- \( S \subseteq V \) is orthogonal if \( \langle x, y \rangle = 0 \) for all distinct \( x, y \in S \).
- If all \( x \in S \) are unit too, then \( S \) is called orthonormal.

**Theorem** Suppose \( S = \{ u_1, \ldots, u_k \} \subseteq V \) is orthonormal. If \( y \in \text{span}(S) \), then \( y = \sum_{i=1}^{k} \langle y, u_i \rangle u_i \).
Moreover, \( S \) is linearly independent.

**Theorem**: Suppose \( V \) is a finite dimensional inner product space. Then \( V \) has an orthonormal basis.

Today, we’ll prove this via an algorithm that builds such a basis.

**Gram-Schmidt Process**

Suppose, \( \{ w_1, w_2, \ldots, w_n \} \subseteq V \) is linearly independent.

Set, \( \hat{v}_1 = w_1 \), \( u_1 = \frac{\hat{v}_1}{\| \hat{v}_1 \|} \).
\( \hat{v}_2 = w_2 - \langle w_2, u_1 \rangle u_1 \), \( u_2 = \frac{\hat{v}_2}{\| \hat{v}_2 \|} \).
\( \vdots \)
\( \hat{v}_k = w_k - \sum_{i=1}^{k-1} \langle w_k, u_i \rangle u_i \), \( u_k = \frac{\hat{v}_k}{\| \hat{v}_k \|} \).

**Claim** \( \{ u_1, \ldots, u_n \} \) is orthonormal with the same span as \( \{ w_1, \ldots, w_n \} \).
Note: Taking \{w_1, \ldots, w_n\} to be a basis of V, proves the thm.

pf. of claim: Induction on n.

Base case: (n = 1). As \text{span} \{u_1\} = \text{span} \{v_1\} and any single unit vector is orthonormal, the claim holds.

Induction step: Suppose that \{v_1, v_2, \ldots, v_{n-1}\} is orthonormal and \text{span} (\{v_1, \ldots, v_{n-1}\}) = \text{span} (\{w_1, \ldots, w_{n-1}\}).

We show that,

\begin{itemize}
  \item[i)] \hat{v}_n \text{ is orthogonal to } \{v_1, v_2, \ldots, v_{n-1}\}
  
  \text{and so is } v_n.
\end{itemize}

This follows directly, as \{v_1, \ldots, v_{n-1}\} is orthonormal and

\[
\langle \hat{v}_n, v_j \rangle = \langle w_n, v_j \rangle - \sum_{i=1}^{n-1} \langle w_n, v_i \rangle \langle v_i, v_j \rangle = 0 \text{ if } i \neq j \text{ and } 1 \text{ if } i = j
\]

\[
= \langle w_n, v_j \rangle - \langle w_n, v_j \rangle = 0 \text{ for all } j < n.
\]

\begin{itemize}
  \item[ii)] \hat{v}_n \in \text{span} \{v_1, \ldots, v_{n-1}, w_n\}
  
  = \text{span} \{w_1, \ldots, w_n\} \text{ by induction}
\end{itemize}

As \{v_1, \ldots, v_n\} is linearly indep, we must have \text{span} \{v_1, \ldots, v_n\} = \text{span} \{w_1, \ldots, w_n\} for dimension reasons.
If we know \( v_3 \), then

\[
\omega_3 = \langle \omega_3, v_1 \rangle v_1 + \langle \omega_3, v_2 \rangle v_2 + \langle \omega_3, v_3 \rangle v_3
\]

can calculate without solve for unit \( v_3 \)!

Knowing \( v_3 

\text{Ex: } V = P_2(\mathbb{R}), \quad \langle f, g \rangle = \int_{-1}^{1} f(x) g(x) \, dx

\{ 1, x, x^2 \} \xrightarrow{\text{Gram Schmidt}} \{ \frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}} x, \sqrt{\frac{5}{8}} (3x^2 - 1) \}

\langle 1, 1 \rangle = \int_{-1}^{1} 1 \cdot 1 \, dx = 2 \quad \Rightarrow \quad \| 1 \| = \sqrt{2}

These are called Legendre polynomials, the next few are

\[ \sqrt{\frac{7}{8}} (5x^3 - 3x), \quad \sqrt{\frac{9}{128}} (35x^4 - 30x^2 + 3), \ldots \]

First appeared in the study of series expansions of gravitational potential for in spherical co-ordinate.
Suppose \( S \subseteq V \) is orthonormal. For \( x \in V \), the scalars \( \langle x, u \rangle \) with \( u \in S \) are called the **Fourier coefficients** of \( x \) relative to \( S \).

Reason for name: \( V = C([-1,1]) \)

and \( \langle f, g \rangle = \int_{-1}^{1} f(x) g(x) \, dx \)

Then, \( S = \left\{ \frac{1}{\sqrt{2}} \right\} \cup \left\{ \sin(\pi n x), \cos(\pi n x) \right\}_{n=1}^{\infty} \) is an orthonormal subset of \( V \).

Consider, \( h(x) = |x| \) in \( V \).

The Fourier coefficients are

\[
\langle h, \frac{1}{\sqrt{2}} \rangle = \int_{-1}^{1} |x| \cdot \frac{1}{\sqrt{2}} \, dx = \frac{\sqrt{2}}{2} \int_{0}^{1} x \, dx = \frac{1}{\sqrt{2}}
\]

\[
\langle h, \sin(\pi n x) \rangle = 0 \quad \text{by symmetry}
\]

\[
\langle h, \cos(\pi n x) \rangle = 2 \int_{0}^{1} x \cos(\pi n x) \, dx = \begin{cases} 0 & n \text{ even} \\ -\frac{4}{n^2 \pi^2} & n \text{ odd} \end{cases}
\]

\( S \) is not a basis of \( V \) and infinitely many of these Fourier coefficients are non-zero.
However, if we allow infinite sums, we get

\[ |x| = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1}{n^2} \cos \pi n x, \quad x \in [-1,1] \]

**Cor:** \( \frac{\pi^2}{8} = \sum_{n=1}^{\infty} \frac{1}{n^2} \), take \( x = 0 \)

**De:** Suppose \( S \subseteq V \) is non-empty. Then

\[ S^\perp = \{ x \in V \mid \langle x, y \rangle = 0 \text{ for all } y \in S \} \]

is called the **orthogonal complement** of \( S \).

**Ex:** \((\mathbb{R}^2, \text{dot prod})\)

\[ S_1 = \{ e_1 \}, \quad S_1^\perp = \text{span}(\{e_1\}) \]

\[ S_2 = \text{span}\{ (1,1) \}, \quad S_2^\perp = \text{span}\{ (-1,1) \} \]

**Ex:** \((\mathbb{R}^3, \text{dot prod})\)

\[ S_1 = \{ e_1 \}, \quad S_1^\perp = \text{xy plane} \]

\[ S_2 = \{ e_1, e_2 \}, \quad S_2^\perp = \text{y-axis} \]

Note, \( S^\perp \) is always a subspace.