Elementary matrices & the Determinant

Math 416 - E13/F13
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Last time:

A ∈ \( M_{n \times n}(\mathbb{R}) \)

1. \( A \xrightarrow{Rr \leftrightarrow R_2} B \) \( \Rightarrow \) \( \det B = -\det A \)

2. \( A \xrightarrow{cRr} B \) \( \Rightarrow \) \( \det B = c \det A \)

3. \( A \xrightarrow{Rr + cR_s} B \) \( \Rightarrow \) \( \det B = \det A \).

Elementary matrices: Result of doing a single row op to \( In \).

Today:

\( \det AB = \det A \cdot \det B \).

Strategy: Relate row ops to matrix multiplications.

Recall that, \( \text{rank}(A) = \dim(\text{Col Sp}(A)) = \dim(\text{Row Sp}(A)) \)

Thm:

For \( A ∈ M_{n \times n}(\mathbb{R}) \), if \( \text{rank}(A) < n \), then \( \det A = 0 \).

pf:

As \( \text{rank}(A) < n \), some row is a linear combination of the others, say

\[ a_r = c_1 a_1 + \cdots + c_{r-1} a_{r-1} + c_{r+1} a_{r+1} + \cdots + c_n a_n \]

where \( a_i \) is the \( i \)th row of \( A \).

If we use row ops \( -c_i R_i + k_i \) for \( i = 1, 2, \ldots, n-1, n \),
then we get a matrix \( B \) whose \( n \)th row is 0. Hence, \( \det B = 0 \) \( \Rightarrow \) \( \det A = 0 \).
**Thm:** Suppose $E$ is the elementary matrix where $I_n \xrightarrow{R} E$. If $A \in M_{n \times n}(\mathbb{R})$, then $A \xrightarrow{R} E A$.

**Example:**

- $R_1 \leftrightarrow R_2$
  
  \[
  \begin{pmatrix}
  0 & 1 \\
  1 & 0
  \end{pmatrix} \begin{pmatrix}
  1 & 2 \\
  3 & 4
  \end{pmatrix} = \begin{pmatrix}
  3 & 4 \\
  1 & 2
  \end{pmatrix}
  \]

- $3R_1$
  
  \[
  \begin{pmatrix}
  3 & 0 \\
  0 & 1
  \end{pmatrix} \begin{pmatrix}
  1 & 2 \\
  3 & 4
  \end{pmatrix} = \begin{pmatrix}
  3 & 6 \\
  3 & 4
  \end{pmatrix}
  \]

- $-R_1 + R_2$
  
  \[
  \begin{pmatrix}
  -1 & 0 \\
  1 & 0
  \end{pmatrix} \begin{pmatrix}
  1 & 2 \\
  3 & 4
  \end{pmatrix} = \begin{pmatrix}
  1 & 2 \\
  2 & 2
  \end{pmatrix}
  \]

**pf of Thm:**

Exc. Prove for all 3 types of row ops.

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**Thm:** Every elementary matrix is invertible.

**pf:**

Suppose $I_n \xrightarrow{R} E$. Let $R'$ be the row op that reverses $R$, that is $A \xrightarrow{R} B \xrightarrow{R'} A$ for all $A \in M_{n \times n}(\mathbb{R})$. Qn: Why does $R'$ exist?

Let $E'$ be the elementary matrix corr. with $R'$.

By prev thm, we have

- $E' E = \text{result of doing } R' \text{ to } E = I_n$
- $EE' = \text{do } R \text{ to } E' = I_n$

So, $E$ is invertible with inverse $E'$. qed
**Thm:** \( A \in M_{n \times n}(\mathbb{R}) \) is invertible if and only if it is the product of elementary matrices.

**Proof:**

\( \Leftarrow \) If \( A = E_1 E_2 \cdots E_k \) with \( E_k \) elementary \( \forall k \), then each \( E_k \) is invertible and so
\[
A^{-1} = E_k^{-1} \cdots E_2^{-1} E_1^{-1}.
\]

\( \Rightarrow \) If \( A \) is invertible, then
\[
B = (A \mid I_n) \xrightarrow[\text{row ops}]{\text{new}} (I_n \mid A^{-1}) = C
\]
As each row op can be implemented by mult by an elementary matrix, we have \( E_1, E_2, \ldots, E_k \) such that
\[
E_k \cdots E_2 E_1 B = C
\]
which implies \( E_k \cdots E_2 E_1 A = I_n \)

\& so, \( A = E_k^{-1} \cdots E_2^{-1} E_1^{-1} \)

As the \( E_k^{-1} \) are also elementary, we're done. \( \Box \)

**Thm:** \( \det(AB) = \det(A) \cdot \det(B) \)

**Proof:** If \( \text{rank}(AB) < n \), then \( \det(AB) = 0 \).
Moreover, one of \( A, B \) must have rank \( < n \) and so one of \( \det A, \det B \) is 0. Thus, in this case, \( \det(AB) = \det A \cdot \det B \).

Thus, now we can assume that \( A, B, AB \) all have rank \( n \).
In particular, \( A = E_1 \cdots E_k, \ B = E_{k+1} \cdots E_m \)
where \( E_k \)’s are elementary.
The result now follows from:

Claim: Suppose \( C = E_1' \ldots E_p' \) where \( E_k' \) are elementary. Then,
\[
det C = (-1)^{\# \text{ type } \circ E_k'} \left( \text{product of } E_k \text{ in all type } \circ E_k' \right)
\]

pf of claim:
\( C \) is obtained from \( I_n \) which has \( det I \), by the row ops \( R_1', \ldots, R_k, R_l' \). By last time, only the type 0 & 2 ops change the \( det \) and do so in a way that proves the claim.

For all elementary matrices \( E \), \( det E^t = det E \).

pf: For type 1 and 2, \( E^t = E \).
For type 3, \( (E_{R_n+cR_s})^t = E_{R_s+cR_n} \)
\[(AB)^t = B^t A^t\]

**Proof:** Check using definition.

**Thm:** For \( A \in M_{m \times n}(\mathbb{R}) \), \( \det A^t = \det A \).

**Proof:** \( \text{Row } sp(A^t) = \text{Col } sp(A) \).

Thus, if \( \det A = 0 \) \( \Rightarrow \) rank \( A < n \)

\[ \Rightarrow \dim \text{Row } sp(A) < n \]

\[ \Rightarrow \dim \text{Col } sp(A^t) < n \Rightarrow \text{rank } A^t < n \Rightarrow \det A^t = 0. \]

If \( A \) is invertible, \( A = E_1 E_2 \cdots E_k \), product of elementary matrices.

Thus,

\[ \det A^t = \det (E_k^t \cdots E_2^t E_1^t) = \prod \det E_k^t \]

\[ = \prod \det E_k = \det A. \]

qed.