Last time: \[ \det : M_{n \times n}(\mathbb{R}) \rightarrow \mathbb{R} \text{ is such that} \]

a) \( \det A \neq 0 \) iff \( A \) is invertible.

b) \( \det AB = \det A \det B \)

c) \( L_A : \mathbb{R}^n \rightarrow \mathbb{R}^n \) changes volume by a factor of \( \det A \)

d) Except for \( n = 1 \), \( \det \) is not linear, but \( n \)-multilinear.

For an \( n \times n \) matrix \( A \), let \( \tilde{A}_{ij} \) denote the \((n-1) \times (n-1)\) matrix obtained by deleting the \( i \)th row & \( j \)th column of \( A \).

\[ \text{For } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \det A = ad - bc \]

For \( n = 2 \),

\[ A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 2 \\ 2 & 1 & 1 \end{pmatrix}, \quad \tilde{A}_{11} = \begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix}, \quad \tilde{A}_{23} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \]

\[ \tilde{A}_{31} = \begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix} \]

**Defn.** For \( A \in M_{n \times n}(\mathbb{R}) \), set \( \det A = A_{11} \).

For \( A \in M_{n \times n}(\mathbb{R}) \) with \( n > 1 \), inductively define,

\[ \det (A) = \sum_{j=1}^{n} (-1)^{1+j} A_{1j} \det (\tilde{A}_{1j}) \]

**Note:** Matches old defn. for \( n = 2 \) as for \( A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \)

we have \( \det A = A_{11} \underbrace{\det (\tilde{A}_{11})}_{A_{11}} - A_{12} \underbrace{\det (\tilde{A}_{12})}_{A_{21}} \)
ex. \[ \det \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 2 \\ 2 & 1 & 1 \end{pmatrix} = 1 \det \begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix} - 2 \det \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} + 3 \det \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \]
\[ = 1 \cdot (-2) - 2 \cdot (1-4) + 3 \cdot (1-0) = 7 \]

ex. \[ \det \begin{pmatrix} 4 & 3 & 2 & 1 \\ 3 & 1 & 2 & 3 \\ 2 & 1 & 0 & 2 \\ 1 & 2 & 1 & 1 \end{pmatrix} = 4 \det \begin{pmatrix} A_{11} \\ A_{12} \\ A_{13} \\ A_{14} \end{pmatrix} - 3 \det \begin{pmatrix} A_{21} \\ A_{22} \\ A_{23} \\ A_{24} \end{pmatrix} + 2 \cdots \]
\[ = 21 \]

This method needs \( n! \) multiplications to compute an \( n \times n \) det. Will develop faster method using our trusty row ops later!

Thm. \( \det \text{ is a linear function of the } r \text{th row when all other rows are held fixed}. \) In particular, suppose \( A, B, C \in M_{n \times n}(\mathbb{R}) \) are the same except in row \( r \), where \( a_r = b_r + k \cdot c_r \) for some \( k \in \mathbb{R} \).

(Here, \( a_r \) is the \( r \)-th row of \( A \).

Then, \( \det A = \det B + k \cdot \det C \).

\( A \) How does this not violate \( 4 \) ?

\( B \) First suppose \( r = 1 \). Then, by defn. of determinant
\[ \det A = \sum_{j=1}^{n} (-1)^{1+j} A_{ij} \det (\tilde{A}_{ij}) \]

\[ = \sum_{j=1}^{n} (-1)^{1+j} (B_{ij} + kC_{ij}) \cdot \det (\tilde{B}_{ij}) \ < \text{also same as} \]

\[ = \sum_{j=1}^{n} (-1)^{1+j} B_{ij} \det (\tilde{B}_{ij}) + k \sum_{j=1}^{n} (-1)^{1+j} C_{ij} \det (\tilde{C}_{ij}) \]

\[ = \det B + k \det C \]

In general, we induct on \( n \).

Base case: \( n = 1 \) where \( \det \) is actually linear.

**Inductive step:** Assume, proven for \( M_{n \times n}(\mathbb{R}) \).

If \( n = 1 \), we are done by above. So, assume \( n > 1 \).

\[ \det A = \sum_{j=1}^{n+1} (-1)^{1+j} A_{ij} \det (\tilde{A}_{ij}) \]

\[ = B_{ij} + C_{ij} \]

\[ = \sum_{j=1}^{n+1} (-1)^{1+j} A_{ij} (\det (\tilde{B}_{ij}) + k \det (\tilde{C}_{ij})) \]

\[ = \det B + k \det C \]
Thm. Suppose $A \in M_{n \times n}(\mathbb{R})$. For any $r$ with $1 \leq r \leq n$, we have,

$$\det A = \sum_{j=1}^{n} (-1)^{r+j} A_{rj} \det \tilde{A}_{rj}$$

Lem. Suppose $B \in M_{n \times n}(\mathbb{R})$ where row $r$ of $B$ equal to $e_j$. Then, $\det B = (-1)^{r+j} \det (B_{rj})$.

Note: When $n=1$, we have $r=j=1$ & $B = (1)$. So for this to hold, let’s define $\det (0 \times 0 \text{ matrix}) = 1$.

pf of thm assuming lemma:

Set $B_j \in M_{n \times n}(\mathbb{R})$ to be $A$ with the $r$-th row replaced by $e_j$. Thus,

$r$-th row of $A = \sum_{j=1}^{n} A_{rj}$ (r-th row of $B_j$)

By the first thm, we have

$$\det A = \sum_{j=1}^{n} A_{rj} \cdot \det (B_j)$$

$$= \sum_{j=1}^{n} A_{rj} \cdot (-1)^{r+j} \det (\tilde{B}_{rj}) \cdot \tilde{A}_{rj}$$
Example:

\[ A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 2 \\ 2 & 1 & 1 \end{pmatrix}, \quad r = 2 \]

\[ \det A = -A_{12} \cdot \det \tilde{A}_{12} + A_{32} \cdot \det \tilde{A}_{32} - A_{23} \cdot \det \tilde{A}_{23} \]

\[ = -1 \cdot \det \begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix} + 0 - 2 \cdot \det \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \]

\[ = -1(-1) - 2(-3) = 7 \]

Proof of Lemma: Pf by induction on n and similar to the first theorem. Gets a bit messy!