Let $V$ and $W$ be vector spaces.

Recall that $I_v: V \to V$ given by $I_v(v) = v$ is the identity transformation on $V$.

Similarly, $I_w: W \to W$ is given by $I_w(w) = w$.

**Defn.** Suppose $T: V \to W$ is linear. A function $S: W \to V$ is an inverse to $T$ if $S \circ T = I_v$ and $T \circ S = I_w$.

**Thm.** Suppose $T: V \to W$ is linear.

1. $T$ has an inverse iff it is 1-1 and onto.
2. If $T$ has an inverse, it is unique and denoted $T^{-1}: W \to V$.
3. If $T^{-1}$ exists, it is linear.

**Proof:** 1. and 2. are standard facts about functions between sets (see [Fis, appendix B]).

For 3., suppose $w_1, w_2 \in W$ and $c \in \mathbb{R}$.

Let $v_1, v_2$ be the unique elements in $V$ with $T(v_1) = w_1$.

Then, $T(cv_1 + v_2) = cT(v_1) + T(v_2) = cw_1 + w_2$.

Hence, $T^{-1}(cw_1 + w_2) = cv_1 + v_2 = cT'(w_1) + T'(w_2)$.

So, $T^{-1}$ is linear.

**Example:** $T: \mathbb{R}^2 \to \mathbb{R}^2$ notation by $\pi/2$ counter-clockwise.

$S: \mathbb{R}^2 \to \mathbb{R}^2$ notation by $\pi/2$ clockwise.

$T: \mathbb{R}^2 \to \mathbb{R}^2$,

$S: \mathbb{R}^2 \to \mathbb{R}^2$

$T: \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} y \\ -x \end{pmatrix}$

$S: \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} y \\ -x \end{pmatrix}$

$T \circ S (x,y) = T( S(x,y) ) = T( y, -x ) = (x, y) = I_{\mathbb{R}^2}(x,y)$

$S \circ T (x,y) = S( T(x,y) ) = S(-y, x) = (x, y) = I_{\mathbb{R}^2}(x,y)$
If \( p = \{ e_1, e_2 \} \), then \( [T]_\beta = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \), \( [s]_\beta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \).

**Thm.** Suppose \( T : V \to W \) is linear and \( \beta \) is a basis for \( V \). If \( T \) is invertible, then 
\[
\gamma = T\beta = \{ T(v) \mid v \in \beta \}
\]
is a basis for \( W \).

**Cor.** If \( V \) is finite-dimensional and \( T : V \to W \) is invertible, then \( W \) is also finite-dimensional with 
\[
\dim W = \dim V.
\]

**pf:** We'll prove the theorem only when \( \beta \) is finite, say \( \beta = \{ v_1, v_2, \ldots, v_n \} \).

Set \( \omega_i = T(v_i) \) so that \( \gamma = \{ \omega_1, \ldots, \omega_n \} \).

1. \( \gamma \) is a spanning set: Let \( w \in W \). There are unique scalars so that,
\[
T^{-1}(w) = a_1 v_1 + a_2 v_2 + \ldots + a_n v_n.
\]
Then,
\[
w = \sum \omega_i = T(T^{-1}(w)) = T(a_1 v_1 + a_2 v_2 + \ldots + a_n v_n) = \sum a_i \omega_i
\]
as needed.

2. \( \gamma \) is linearly independent: Suppose that,
\[
a_1 \omega_1 + a_2 \omega_2 + \ldots + a_n \omega_n = 0.
\]
Then,
\[
0 = T^{-1}(0) = a_1 T^{-1}(\omega_1) + \ldots + a_n T^{-1}(\omega_n)
= a_1 v_1 + \ldots + a_n v_n
\]
\[
\Rightarrow a_i = 0 \forall i \quad \text{as } \beta \text{ is linearly indep}
\]
so, \( \gamma \) is linearly indep.
Deb: Vector spaces $V$ and $W$ are isomorphic if there exists an invertible linear transformation $T: V \to W$. Such a $T$ is called an isomorphism.

Ex: $T: \mathbb{R}^3 \to \mathbb{P}_2(\mathbb{R})$ where $T(a, b, c) = a + bx + cx^2$

Thm: Suppose that $V$ is finite dimensional. Then, a vector space $W$ is isomorphic to $V$ iff $W$ is finite dimensional and $\dim W = \dim V$.

Pf: If $V$ and $W$ are isomorphic, we've already shown that $\dim W = \dim V$.

For the converse, take two bases $\beta = \{v_1, \ldots, v_n\}$ and $\gamma = \{w_1, \ldots, w_n\}$ of $V, W$, respectively.

Define, $T: V \to W$ such that $[T]_{\beta}^{\gamma} = I_n$, i.e. $T(v_i) = w_i \quad \forall i = 1, 2, \ldots, n$.

Clearly, $T$ is uniquely determined by its value on $\beta$. Check that $T$ is invertible and hence an isomorphism.

Note that, isomorphism is an equivalence relation.
**Definition:** An $n \times n$ matrix $A$ is invertible if there exists an $n \times n$ matrix $B$ with $AB = BA = I_n$.

**Example:**

\[
A = \begin{pmatrix} 1 & 1 \\ -2 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix}
\]

\[
AB = \begin{pmatrix} 1 & 1 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I
\]

\[
BA = \begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I
\]

Note that, if $A$ has an inverse, it is unique. If $C$ also has $AC = CA = I_n$, then $C = C I_n = CAB = B$.

When it exists, the inverse of $A$ is denoted by $A^{-1}$.

**Connections**

Suppose $T : V \rightarrow W$ is an isomorphism between vector spaces of dim $n$. Fix two bases $\beta, \gamma$ of $V, W$, respectively. If

\[ A = [T]_{\gamma}^\beta, \quad B = [T^{-1}]_{\gamma}^\beta \]

then $B = A^{-1}$.

**Proof:**

\[
AB = [T]_{\gamma}^\beta [T^{-1}]_{\gamma}^\beta = [T \circ T^{-1}]_{\gamma}^\beta = [I_W]_{\gamma}^\beta = I_n
\]

Similarly, $BA = [T^{-1}]_{\gamma}^\beta [T]_{\gamma}^\beta = [T^{-1} \circ T]_{\gamma}^\beta = [I_V]_{\gamma}^\beta = I_n$. 