Last class: V, W vector spaces. A function $T: V \to W$ is a linear transformation (or linear) if for all $v_1, v_2 \in V$ and $c \in \mathbb{R}$ we have

1) $T(v_1 + v_2) = T(v_1) + T(v_2)$

2) $T(cv) = c \cdot T(v)$

Ex: $T: \mathbb{R}^2 \to \mathbb{R}^2$

$T(x, y) = (-y, x)$

Rotation by $\pi/2$ counterclockwise

Ex: $S: \mathbb{R}^2 \to \mathbb{R}^2$

$S(x, y) = (x+y, y)$
Ex.

\[ T: \mathbb{C}[0,1] \rightarrow \mathbb{R} \]

\[ \{ f: [0,1] \rightarrow \mathbb{R} \mid f \text{ is continuous} \} \]

\[ T(f) = \int_0^1 f'(x) \, dx, \quad T(x) = \frac{1}{2}, \quad T(x^2) = \frac{1}{3} \]

\[ T \text{ is linear is the basic properties of definite integrals.} \]
Suppose $T: V \to W$ is linear. If $\beta = \{v_1, v_2, \ldots, v_n\}$ is a basis for $V$ then $T$ is determined by its values on $\beta$. Moreover,

$$R(T) = \text{span} \{ T(v_1), T(v_2), \ldots, T(v_n) \}$$

**pf.** Suppose we know $T(v_1), \ldots, T(v_n)$. Given $v \in V$, there are unique scalars such that

$$v = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n.$$ 

Repeatedly using properties i) & ii) for linear trans. we get,

$$T(v) = a_1 T(v_1) + a_2 T(v_2) + \cdots + a_n T(v_n).$$

So, \{ $T(v_i)$, $i=1,2,\ldots,n$ \} determine $T$. 
Also, we have $R(T) \subseteq \text{span} \{ T(v_i), i=1,2,\ldots,n \}$.

As, $T(v_i) \in R(T)$ for all $i$ and $R(T)$ is a subspace, we have $R(T) \supseteq \text{span} \{ T(v_i), i=1,2,\ldots,n \}$.

Thus,

$$R(T) = \text{span} \{ T(v_i), i=1,2,\ldots,n \}$$

as claimed.

Recall that, $N(T) = \{ v \in V \mid T(v) = 0 \}$ is also a subspace.
Dimension Thm: Suppose that \( T : V \rightarrow W \) is linear and \( V \) is finite \( \text{dim} \ V \). Then,

\[
\text{dim } N(T) + \text{dim } R(T) = \text{dim } V
\]

Nullity of \( T \) Range of \( T \)

More examples, \( T : \mathbb{R}^{a+b} \rightarrow \mathbb{R}^{a+c} \)

\[
(x_1, \ldots, x_{a+b}) \mapsto (x_1, \ldots, x_a, 0, \ldots, 0)
\]

Here, \( R(T) = \left\{ (x_1, \ldots, x_a, 0, 0, \ldots, 0) \right\} \mid x_1, \ldots, x_a \in \mathbb{R} \}

and \( N(T) = \left\{ (0, \ldots, 0, x_{a+1}, \ldots, x_{a+b}) \right\} \mid x_{a+1}, \ldots, x_{a+b} \in \mathbb{R} \}

So, \( \text{nullity} + \text{rank} = b + a = \text{dim } \mathbb{R}^{a+b} \).
Ex: \[ \mathbb{R}^3 \rightarrow \mathbb{R}^2 \]
\[ (x, y, z) \mapsto (x, y) \]

\[ \mathbb{R}^3 \rightarrow \mathbb{R} \]
\[ (x, y, z) \mapsto x \]

pf of the dimension thm:

Idea: In the right co-ordinates (= bases) any linear transformation looks like these examples.

Let \( \beta' \) be a basis for \( N(T) \). By the replacement thm, we can enlarge \( \beta' \) to a basis \( \beta \) of \( V \), say...
\[ \beta = \{ v_1, \ldots, v_a, \varphi_1, \ldots, \varphi_b \} \quad \text{with} \]
\[ \beta' = \{ v_{a+1}, \ldots, v_{a+b} \} \]

It is enough to show that
\[ \gamma = \{ T(v_1), \ldots, T(v_a) \} \]
is linearly independent as then by the last theorem it is a basis for \( \mathbb{R}(T) \) and so,
\[ \text{nullity} + \text{rank} = b + 181 = b + a = |\beta| = \dim V. \]

Suppose, \( c_1 T(v_1) + c_2 T(v_2) + \ldots + c_a T(v_a) = 0 \)
By linearity of $T$, we have

$$T(c_1v_1 + \cdots + c_nv_n) = 0$$

or

$$u = c_1v_1 + \cdots + c_nv_n \in N(T),$$

Thus, $w$ is a linear combination of $\beta'$ with

$$w = c_{a_1}v_{a_1} + \cdots + c_{a+b}v_{a+b}.$$

By linear indep of $\beta$, we must have all $c_i \equiv 0$.

So, $Y$ is linearly independent as needed to prove the thm.