We have studied vector spaces and subspaces.

Now, we look at functions between them.

Ex.

\[ f : \mathbb{R}^2 \rightarrow \mathbb{R}^3 \]

\[ f(s, t) = (s, \cos t, \sin t) \]

In this class, we'll focus on the simplest kind of functions between vector spaces, e.g.

Ex.

\[ f : \mathbb{R}^2 \rightarrow \mathbb{R}^3 \]

\[ f(s, t) = (s, t, s + t) \]
**Defn.** Suppose that $V$ and $W$ are vector spaces over $\mathbb{R}$.

A fn $T : V \rightarrow W$ is a **linear transformation** if for all $v_1, v_2 \in V$ and $a \in \mathbb{R}$, we have

i) $T(v_1 + v_2) = T(v_1) + T(v_2)$

ii) $T(au) = a \cdot T(v)$

**Ex.** $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$

$T(x, y) = (x - y, x + y)$

Check: $T$ is a linear transformation

$v_1 = (x_1, y_1), \ v_2 = (x_2, y_2)$

$T(v_1 + v_2) = \ldots$

Scalar mult is similar.
By prop ii), namely \( T(\alpha v) = \alpha T(v) \), we see that \( T \) must send a line through 0 "to another one. In fact, this is true for any line in \( \mathbb{R}^2 \) as

\[
T(b_0 + t v_0) = T(b_0) + t \cdot T(v_0) \quad \forall t \in \mathbb{R}.
\]

**Ex.**

\[
T : \mathbb{R}^2 \to \mathbb{R}^3 \quad T(x, y) = (x, y, x+y)
\]

**Ex.**

\[
T : \mathbb{R}^3 \to \mathbb{R}^2 \quad T(x, y, z) = (x, y)
\]

**Ex.**

\[
T : \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})
\]

\[
T(f) = f'
\]

\[
T(x^3 + 2x + 1) = 3x^2 + 2
\]

The fact that this is a linear transformation is one of the first things you learned about derivatives in Calc I.
Ex. \[ T : M_{m \times n}(\mathbb{R}) \rightarrow M_{n \times m}(\mathbb{R}) \]
\[ A \rightarrow A^t \]

Defn. Suppose \( T : V \rightarrow W \) is a linear transformation.

The **nullspace (kernel)** of \( T \) is
\[ N(T) = \{ v \in V \mid T(v) = 0 \} \]

The **range (image)** of \( T \) is
\[ R(T) = \{ T(v) \mid v \in V \} \]
Then these are subspaces of $V$ and $W$, respectively.

**Proof**: For $\mathcal{N}(T)$:

i) $T(0_w) = T(0 \cdot 0_v) = 0 \cdot T(0_v) = 0_w$

so $0_v \in \mathcal{N}(T)$

ii) Suppose $v_1, v_2 \in \mathcal{N}(T)$. Then

$T(v_1 + v_2) = T(v_1) + T(v_2) = 0_w + 0_w = 0_w$

iii) Suppose $a \in \mathbb{R}$ and $v_1 \in \mathcal{N}(T)$. Then

$T(a \cdot v_1) = a \cdot T(v_1) = a \cdot 0_w = 0_w$

and so $a \cdot v_1 \in \mathcal{N}(T)$.

The case of $\mathcal{R}(T)$ is similar.
Examples:

\[ T : \mathbb{R}^3 \to \mathbb{R}^2 \quad T(x, y, z) = (x, y) \]

\[ N(T) = \]

\[ R(T) = \]

Dimension Thm: Suppose that \( T : V \to W \) is a linear transformation. If \( V \) is finite dimensional, then

\[ \dim N(T) + \dim R(T) = \dim V. \]

Nullity

Rank.
Encoding a linear transformation.

Suppose, \( T: V \rightarrow W \) is a linear transformation.

Suppose, \( \beta = \{ v_1, v_2, \ldots, v_n \} \) is a basis for \( V \).

Knowing \( T(v_i) \), \( i = 1, 2, \ldots, n \) determines \( T \) since by repeatedly applying the linearity property we get

\[
T(\sum b_i v_i) = \sum b_i T(v_i) = \sum b_i (v_i) = \sum b_i (v_i)
\]

If \( \gamma = \{ w_1, w_2, \ldots, w_m \} \) is a basis for \( W \), we can write

\[
T(v_j) = \sum_1^m a_{ij} w_i + \sum_1^m a_{ij} w_i + \ldots + \sum_1^m a_{ij} w_i
\]
Hence, we can encode $T$ completely by the matrix

$$[T]_{\beta}^\varphi = \left( (a_{ij}) \right) \in M_{m \times n}(\mathbb{R})$$

**Example.**

$T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$, $T(x, y) = (x + 2y, 3x + 4y)$

$\beta = \varphi = \{ (1,0), (0,1) \}$

$e_1, e_2$

$T(e_1) = (1, 3) = e_1 + 3e_2$

$T(e_2) = (2, 4) = 2e_1 + 4e_2$

$$[T]_{\beta}^\varphi = \left( \begin{array}{cc} 1 & 2 \\ 3 & 4 \end{array} \right)$$

More examples...