Suppose \( W \) is a subspace of a vector space \( V \).

How do we answer the following questions?

1. Given \( w_1, \ldots, w_k \in W \), is \( \text{span} \{ w_1, \ldots, w_k \} = W \)?
2. What is the smallest number of such \( w_i \)'s needed to span \( W \)? How can we find such a set?
3. Given \( u \in V \), is \( u \in W \)?
4. In \( \mathbb{R}^3 \), we've seen 4 kinds of subspaces:
   i) \( \{ 0 \} \)
   ii) \( \mathbb{R}^3 \)
   iii) lines through \( 0 \)
   iv) planes through \( 0 \)

How do we distinguish them mathematically? Or generalize to \( \mathbb{R}^n \)?
**Definition:** Vectors \( u_1, u_2, \ldots, u_k \) in \( V \) are **linearly dependent** if there are scalars \( a_1, a_2, \ldots, a_k \) not all 0, such that \( a_1 u_1 + \cdots + a_k u_k = 0 \). Otherwise, they are **linearly independent**.

**Example:** \( V = \mathbb{R}^2 \), \( u_1 = (-3, 2) \), \( u_2 = (2, 1) \), \( u_3 = (0, 1) \).

Are these dependent?

**Solve:** 
\[
\begin{align*}
\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 &= 0 \\
\alpha_1 (-3\alpha_1 + 2\alpha_2, 2\alpha_1 + \alpha_2 + \alpha_3) &= 0
\end{align*}
\]

This gives 2 equations in 3 variables:
\[
\begin{align*}
-3\alpha_1 + 2\alpha_2 &= 0 \\
2\alpha_1 + \alpha_2 + \alpha_3 &= 0
\end{align*}
\]
So, we're trying to find the null space of

\[
A = \begin{pmatrix} -3 & 2 & 0 \\ 2 & 1 & 1 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}
\]

which is

\[
\left\{ \begin{pmatrix} 2t \\ 3t \\ -7t \end{pmatrix} \mid t \in \mathbb{R} \right\}.
\]

Thus, the vectors \( u_1, u_2, u_3 \) are linearly dependent.

Linear dependency means there is a redundancy amongst the vectors in terms of the span:

\[
u_3 = \frac{2}{7} u_1 + \frac{3}{7} u_2
\]

\[
\Rightarrow \text{span} \{ u_1, u_2, u_3 \} = \text{span} \{ u_1, u_2 \}.
\]
In contrast, \(a,u_1 + azu_2 = 0\) has only the trivial solution and so,

\(u_1, u_2\) are \underline{linearly independent}.

\[\text{example.}\]

\[V = \mathbb{R}^3, \quad u_1 = (-1, 1, 2), \quad u_2 = (1, 2, 1), \quad u_3 = (5, 1, -4)\]

\[W = \text{span} \{u_1, u_2, u_3\}\]

\[\underline{Q}\]

What is \(W\) geometrically? Plane on \(\mathbb{R}^3\)?

Are these \underline{linearly dependent}? The condition

\[a, u_1 + azu_2 + azu_3 = 0\]

gives rise to eqns

\[
\begin{cases}
-a_1 + a_2 + 5a_3 = 0 \\
a_1 + 2a_2 + a_3 = 0 \\
2a_1 + a_2 - 4a_3 = 0
\end{cases}
\]
Equivalently, we need to find the null space of

$$A = \begin{pmatrix} -1 & 1 & 5 \\ 1 & 2 & 1 \\ 2 & 1 & -4 \end{pmatrix} = \begin{pmatrix} u_1 & u_2 & u_3 \end{pmatrix}$$

$$M = \text{Augmented mat} = \begin{pmatrix} -3 & 1 & 5 & 6 \\ 1 & 2 & 1 & 0 \\ 2 & 1 & -4 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & -5 & 0 \\ 0 & 3 & 6 & 0 \\ 0 & 3 & 6 & 0 \end{pmatrix}$$

$$R_2 - R_1 \rightarrow \begin{pmatrix} 1 & -1 & -5 & 0 \\ 0 & 3 & 6 & 0 \\ 0 & 3 & 6 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & -5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 3 & 6 & 0 \end{pmatrix}$$

$$R_3 - 3R_2 \rightarrow \begin{pmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The solution set is $$\{ (3t, -2t, t) \mid t \in \mathbb{R} \} \rightarrow \text{is a sol.}$$
More generally, we say $S \subseteq V$ is \textbf{linearly dependent} if there are distinct vectors $u_1, u_2, \ldots, u_k$ in $S$ which are linearly dependent; otherwise, call $S$ \textbf{linearly independent}.

\textbf{Notes:}

1. The set $\{0\}$ is \textbf{linearly dependent} as $1 \cdot 0 = 0$.

2. If $u \neq 0$, $\{u\}$ is \textbf{linearly independent} as $a \cdot u = 0 \Rightarrow \frac{1}{a} \cdot au = 0 \Rightarrow u = 0 \Rightarrow a \neq 0$.

3. The empty set is \textbf{linearly independent}.

By convention, $\text{Span}(\emptyset) = \{0\}$. 
**Thm.** Suppose \( u_1, u_2, \ldots, u_k \) are non-zero vectors in \( V \) and consider the subspace
\[
W = \text{span} \left\{ u_1, u_2, \ldots, u_k \right\}.
\]
Then, there exists a linearly independent subset \( u_{i_1}, u_{i_2}, \ldots, u_{i_k} \) of the \( u_i \)'s where
\[
W = \text{span} \left\{ u_{i_1}, u_{i_2}, \ldots, u_{i_k} \right\}.
\]

**pf.** *Induction on \( k \):*

*Base Case:* \( k = 1 \). Done.

*Ind. Step:* Suppose \( u_1, \ldots, u_k \) are non-zero vectors in \( V \) with \( W = \text{span} \{ u_1, \ldots, u_k \} \).
If \( u_1, \ldots, u_k \) are linearly independent, then we're done.

Otherwise, \( u_1, \ldots, u_k \) are linearly dependent.

Then, we can express one of the \( u_i \)'s as a linear comb of the rest.

We can drop the vector from the \( \{u_1, \ldots, u_k\} \) collection having the same span. This new collection has size \( k-1 \).

\[ \text{QED.} \]

**Algorithm.**

1. \( S = \{u_1, \ldots, u_k\} \)
2. check if \( \{u_1, \ldots, u_k\} \) is independent, if yes stop.
3. If not, remove one vector, that can be written as a linear comb of the other.
4. \( \text{Go to 1.} \)
1. What can we do if \( k = \infty \)?

2. Is the size of the smallest independent subset unique?

- \( \text{lin ind} = \) linearly independent

- \( \text{lin dep} = \)