Linear combinations

Math 416 - E13/F13 01/26/2022

Last class: A subspace $W$ of a vector space $V$ is a subset where

a) $0 \in W$; b) $W$ is closed under addition
and c) $W$ is closed under scalar multiplication.

Today's Goal: How to create a subspace?

Defn: A linear combination of vectors $u, u, \ldots, u_n$ in $V$ is any vector of the form

$$a_1 u_1 + a_2 u_2 + \ldots + a_n u_n$$

where $a_1, a_2, \ldots, a_n \in \mathbb{R}$.

Example: $V = \mathbb{R}^3$, $u_1 = (1, -1, 0)$, $u_2 = (0, 0, 1)$

$$(1, -1, 1) = 1 u_1 + 1 u_2$$
$$(0, 0, -1) = 0 u_1 + (-1) u_2$$

Defn: The span of vectors $u, u, \ldots, u_n$ in $V$ is the set of all linear combinations of $u, u, \ldots, u_n$.

Example: $\text{span}(u, u_2) = \left\{ a_1 u_1 + a_2 u_2 \mid a_1, a_2 \in \mathbb{R} \right\}$

$$= \left\{ (a_1, -a_1, a_2) \mid a_1, a_2 \in \mathbb{R} \right\}$$

= plane in $\mathbb{R}^3$ given by $x + y = 0$
**Theorem:** The span of any \( u_1, u_2, \ldots, u_n \) in \( V \) is a subspace of \( V \).

**Pf.** Check 3 conditions of subspace definition:

a) \( 0 = 0u_1 + 0u_2 + \cdots + 0u_n \)

b) \[ (a_1u_1 + a_2u_2 + \cdots + a_nu_n) + (b_1u_1 + b_2u_2 + \cdots + b_nu_n) = (a_1 + b_1)u_1 + (a_2 + b_2)u_2 + \cdots + (a_n + b_n)u_n \]
   
   why?

 c) For any \( c \in \mathbb{R} \),
   
   \[ c \cdot (a_1u_1 + a_2u_2 + \cdots + a_nu_n) = (ca_1)u_1 + (ca_2)u_2 + \cdots + (ca_n)u_n \]

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**Example.** \( V = \mathbb{R}^3 \), \( u_1 = (1,1,-1) \), \( u_2 = (-1,1,2) \)

\( W = \text{span}(u_1,u_2) \)

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**An. 1.** Find eqn for \( W \) of the form

\[ c_1x + c_2y + c_3z = 0 \]

Since \( u_1, u_2 \in W \), we must have

\[
\begin{cases}
  c_1 + c_2 - c_3 = 0 \\
  -c_1 + c_2 + 2c_3 = 0
\end{cases}
\]

add top \( \rightarrow \) to bottom

\[
\begin{cases}
  c_1 + c_2 - c_3 = 0 \\
  2c_2 + c_3 = 0
\end{cases}
\]
add bottom $c_1 + 3c_2 = 0$

$\Rightarrow$

top $2c_2 + c_3 = 0$

If we take $c_2 = -1$, we get $c_1 = 3$, $c_3 = 2$

So eqn for plane is \{ $3x - y + 2z = 0$ \}

Rat Poisson Test:

**Qn 2.**

$v = (3, 1, -4)$ is in $W$ as it satisfies the plane equation. Thus, $v$ must be a linear combination of $u_1$ and $u_2$, i.e.

$v = a_1 u_1 + a_2 u_2$

on $(3, 1, -4) = (a_1 - a_2, a_1 + a_2, -a_1 + 2a_2)$

leading to a system of three equations:

\[
\begin{align*}
  a_1 - a_2 &= 3 \\
  a_1 + a_2 &= 1 \\
  -a_1 + 2a_2 &= -4
\end{align*}
\]

adding we get $2a_1 = 4$

$\Rightarrow$ adding we get $3a_2 = -3$

on $a_1 = 2$

on $a_2 = -1$
Check that, \[ 2 \mathbf{u}_1 - \mathbf{u}_2 = (2, 2, -2) + (1, -1, -2) = (3, 1, -4) \]

Note that, \( \mathbf{w} = (6, 1, -7) \) is NOT in \( \mathbf{W} \).

If we try to write \( \mathbf{w} \) as a linear comb. of \( \mathbf{u}_1 \) and \( \mathbf{u}_2 \), we are lead to

\[
\begin{align*}
    a_1 - a_2 &= 6 & \text{adding} \Rightarrow a_1 = \frac{7}{2} \\
    a_1 + a_2 &= 1 & \text{adding} \Rightarrow a_2 = -2 \\
    -a_1 + 2a_2 &= -7 & \text{adding} \Rightarrow a_2 = -2
\end{align*}
\]

But, \( \frac{7}{2} + (-2) \neq 1 \).

So, there are no solutions to this system, which makes sense geometrically.

### System of linear Equations

**Variables:** \( x_1, x_2, \ldots, x_m \)

**Equations:**

\[ a_{11} x_1 + a_{12} x_2 + \cdots + a_{1m} x_m = b_1, \]
\[ a_{21} x_1 + a_{22} x_2 + \cdots + a_{2m} x_m = b_2, \]
\[ \vdots \]
\[ a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mm} x_m = b_m, \]

where \( a_{ij}, b_i \) are real numbers for \( 1 \leq i \leq n, 1 \leq j \leq n \).

**Why is it called linear?**

No higher order power.

No non-linear \( f(x) \) of \( x \)'s.
How to solve?

- Add two equations together
- Add a multiple of one equation to another

Example. For, \[
\begin{align*}
2x_1 + x_2 - 3x_3 &= 3 \\
3x_1 - 2x_2 + 6x_3 &= 1
\end{align*}
\]

Can take \(2(\text{Eqn 1}) + (\text{Eqn 2})\) to get

\[
7x_1 = 7 \Rightarrow x_1 = 1
\]

and thus, \(x_2 - 3x_3 = 1\)

Q. How to solve such equations in a systematic way?