Last class: Vector Space over \( \mathbb{R} \) is a set with two operations vector addition and scalar multiplication satisfying

1. Vector addition is commutative & associative.
2. There is a zero vector.
3. Existence of additive inverse.
5. Scalar multiplication is associative.
6. Distributive properties.

Examples: \( \mathbb{R}^n \), \( M_{m \times n}(\mathbb{R}) \), \( \mathcal{F}([-1,1]; \mathbb{R}) \).
One more example: (Polynomials of degree ≤ n)

\[ P_n (\mathbb{R}) = \{ a_0 + a_1 x + \ldots + a_n x^n \mid a_0, a_1, \ldots, a_n \in \mathbb{R} \} \]

the set of polynomials with degree ≤ n is a vector sp.

Other Objects: In \( \mathbb{R}^3 \), we have lines and planes.

Today's Goal: Define, similar concepts in general vector spaces.
If $W$ is a plane in $\mathbb{R}^3$ containing 0, $w_1, w_2$ are vectors in $W$, then

- $w_1 + w_2$ is in $W$
- $cw$ is in $W$ for any $c \in \mathbb{R}$

**Note:** It is important that $0 \in W$, otherwise the above conclusion need not be true.

**Definition:** A subset $W$ of a vector space $V$ over $\mathbb{R}$ is a subspace if

i) $0 \in W$
ii) for all $w_1, w_2 \in W$, $w_1 + w_2 \in W$
iii) for all $c \in \mathbb{R}$, $w \in W$, $cw \in W$
Example: Some subspaces in $\mathbb{R}^3$:

i) $\{0\}$ is the smallest subspace.

ii) $\mathbb{R}^3$ is itself a subspace.

iii) Lines $\{t \cdot v \mid t \in \mathbb{R}\}$ on planes $\{ax + by + cz = 0\}$ are subspaces.

Lemma: $\{0\}$ and $\text{V}$ are subspaces of any vector space $\text{V}$. 
Theorem: Let $W$ be a subspace of a vector space $V$. Then, $W$ is a vector space under the operations inherited from $V$.

pf. How to prove? Check the rules for vector spaces.

First, $W$ is closed under vector addition & scalar multiplication by ii) & iii).

- Condition (1) follows from the fact that $V$ is a vector space.
- Condition (3) follows from i)
- Condition (4) follows from iii) as $-v = (-1)u$. 
Non-Example: \( W = \{ (a_1, a_2) : a_i \geq 0 \} \subseteq \mathbb{R}^2 \) is NOT a subspace.

Why? \(-1\) \((a_1, a_2) \notin W \) if \( a_1, a_2 > 0 \).

One more example: For a matrix \( A \in M_{m \times n}(\mathbb{R}) \), we define the transpose as \( A^t \in M_{n \times m}(\mathbb{R}) \) where \( (A^t)_{ij} = A_{ji} \) for \( 1 \leq i \leq n, 1 \leq j \leq m \).

- A matrix \( A \) is called **symmetric** if \( A = A^t \).
- A symmetric matrix is always **square**.
Theorem The subset of symmetric matrices $M_{n \times n}^{sym}(\mathbb{R})$ in $M_{n \times n}(\mathbb{R})$ is a subspace.

pf. Check the 3 conditions of subspace.

i) The $0_{n \times n}$ matrix is symmetric & thus $\in M_{n \times n}^{sym}(\mathbb{R})$

We claim that: $(aA + bB)^t = aA^t + bB^t$ $\forall$ $a, b \in \mathbb{R}$ $A, B \in M_{n \times n}(\mathbb{R})$

Then, ii) and iii) follows as for $A, B \in M_{n \times n}^{sym}(\mathbb{R})$, 

$(A + B)^t = A^t + B^t = A + B$ & $(CA)^t = CA^t = CA$ $\forall C \in \mathbb{R}$.

To prove the claim, note that 

$(aA + bB)^t_{ij} = (aA + bB)_{ji} = aA_{ji} + bB_{ji} = a(A^t)_{ji} + b(B^t)_{ji} = (aA^t + bB^t)_{ij}$

for all $1 \leq i, j \leq n$. □
Another Theorem: Intersection of two subspaces in a vector space, is a subspace.

Why? Check the three conditions of subspaces.

In the book, the definition of subspace is different. But, using our theorem we can see that the definitions are equivalent.