Q1. Let

\[ A = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}. \]

(a) Compute the characteristic polynomial of \( A \).

Solution: \( \text{det}(A - tI_2) = (1-t)(-2-t) - 4 = t^2 + t - 6 = (t-2)(t+3). \)

(b) Find all the eigenvalues of \( A \).

Solution: \( 2, -3 \).

(c) For each eigenvalue compute the corresponding eigenspace.

Solution: \( E_2(A) = \mathcal{N}(A - 2I) = \{ (t, t) \mid t \in \mathbb{R} \}, E_{-3}(A) = \mathcal{N}(A + 3I) = \{ (t, -4t) \mid t \in \mathbb{R} \}. \)

(d) Find an eigenbasis of \( \mathbb{R}^2 \).

Solution: \( \{ (1, 1), (1, -4) \} \)

(e) Now diagonalize \( A \), giving both a diagonal matrix \( D \) and an invertible matrix \( Q \) so that \( A = QDQ^{-1} \).

Solution: \( D = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}, Q = \begin{pmatrix} 1 & 1 \\ 1 & -4 \end{pmatrix} \)

Q2. (a) Consider the parallelepiped \( P \) in \( \mathbb{R}^3 \) determined by the three vectors \( v = (3, 2, 1) \), \( u = (0, 2, 3) \), and \( w = (0, -2, 0) \). Compute the unsigned volume of \( P \).

Solution: \( \text{det} \begin{pmatrix} 3 & 0 & 0 \\ 2 & 2 & -2 \\ 1 & 3 & 0 \end{pmatrix} = 3 \cdot 3 \cdot 2 = 18. \)

(b) Let \( A, B \in \mathcal{M}_{3 \times 3}(\mathbb{R}) \) such that

\[ AB = \begin{pmatrix} 2 & 4 & 0 \\ 1 & 3 & -2 \\ 0 & 0 & 1 \end{pmatrix}. \]

Find \( \text{det}(ABAB) \). Justify each step.

Solution: \( \text{det}(ABAB) = \text{det}(AB)^2 = 4 \) since \( \text{det}(AB) = 2. \)

Q3. Let \( A \in \mathcal{M}_{n \times n}(\mathbb{R}) \) such that \( A^4 = A \).

(a) Prove that the only possible eigenvalues for the matrix \( A \) are 0 and 1.

Solution: Let \( \lambda \in \mathbb{R} \) be an eigenvalue of \( A \) with a corresponding eigenvector \( v \), so that \( Av = \lambda v \). Then \( A^4 = A \) implies that \( \lambda^4 v = A^4 v = \lambda^4 v \) and since \( v \neq 0 \) we have \( \lambda^4 = \lambda \) or \( \lambda \in \{ 0, 1 \} \) since \( \lambda \) is real.

(b) Prove that, \( A \) is diagonalizable iff \( \text{nullity}(A) + \text{nullity}(A - I_n) = n. \)

Solution: Note that \( \text{nullity}(A) = \text{geometric multiplicity of 0} \) and \( \text{nullity}(A - I_n) = \text{geometric multiplicity of 1}. \)

Only if part: Since \( A \) is diagonalizable and the only eigenvalues of \( A \) are 0, 1 we have algebraic multiplicity of 0 + algebraic multiplicity of 1 = \( n \); geometric multiplicity of 0 = algebraic multiplicity of 0, and geometric multiplicity of 1 = algebraic multiplicity of 1, which proves the result.

If part: We break into 3 cases, i) \( \text{nullity}(A) = 0 \), ii) \( \text{nullity}(A_I) = 0 \) and iii) \( \text{nullity}(A) > 0, \text{nullity}(A - I) > 0. \) In all cases we have sum of all algebraic multiplicities = \( n \); geometric
multiplicity of 0 = algebraic multiplicity of 0, and geometric multiplicity of 1 = algebraic multiplicity of 1. Thus the characteristic polynomial of A completely factorizes over \( \mathbb{R} \) and algebraic and geometric multiplicities are the same for all eigenvalues.

Q4. (a) The island of Madagascar is home to almost all of the worlds lemurs, a family of primitive non-ape primates. Divide Madagascar into 2 regions, the North and the South, along the Mania river. Suppose that in any given year, 1/6 of the lemurs living in the North move to the South, and 1/3 of those in the South move to the North. Give the transition matrix \( P \) for the corresponding Markov chain.

\[
P = \begin{pmatrix}
\frac{5}{6} & \frac{1}{3} \\
\frac{1}{6} & \frac{2}{3}
\end{pmatrix}.
\]

**Solution:**

\[
P = \begin{pmatrix}
\frac{5}{6} & \frac{1}{3} \\
\frac{1}{6} & \frac{2}{3}
\end{pmatrix}.
\]

(b) Find a probability vector \( \mathbf{u} \) in \( E_1 \).

**Solution:** We find \( E_1(P) = \{ (2t, t) \mid t \in \mathbb{R} \} \). Thus \( \mathbf{u} = (2/3, 1/3) \).

(c) What is

\[
\lim_{m \to \infty} P^m = \begin{pmatrix}
\frac{2}{3} & \frac{2}{3} \\
\frac{1}{3} & \frac{1}{3}
\end{pmatrix}.
\]

**Solution:**

\[
\lim_{m \to \infty} P^m = \begin{pmatrix}
\frac{2}{3} & \frac{2}{3} \\
\frac{1}{3} & \frac{1}{3}
\end{pmatrix}.
\]

(d) Let \( V \) be an inner product space with norm \( \| \cdot \| \) and \( \mathbf{v}, \mathbf{u} \in V \) be two vectors such that

\[
\| \mathbf{u} \| = 2, \| \mathbf{v} \| = 2, \| \mathbf{u} - \mathbf{v} \| = 3.
\]

Find

\[
\| \mathbf{u} + \mathbf{v} \| = 
\]

**Solution:** Check that \( \| \mathbf{u} + \mathbf{v} \|^2 + \| \mathbf{u} - \mathbf{v} \|^2 = 2 \| \mathbf{u} \|^2 + 2 \| \mathbf{v} \|^2 \). Thus \( \| \mathbf{u} + \mathbf{v} \|^2 = 2(2^2 + 2^2) - 3^2 = 7 \) and \( \| \mathbf{u} + \mathbf{v} \| = \sqrt{7} \).

Q5. Circle true or false as appropriate; you DO NOT need to provide any justification.

(a) A square matrix is diagonalizable if and only if its characteristic polynomial splits completely and it has distinct eigenvalues.

\[ \text{F} \]

(b) If \( E, F \) are elementary matrices of size \( n \times n \), then so is \( EF \).

\[ \text{F} \]

(c) If a \( 4 \times 4 \) matrix has 4 distinct eigenvalues, it is diagonalizable.

\[ \text{- 2 of 3} \]
(d) For $A, B \in \mathcal{M}_{n \times n}(\mathbb{R})$, we have $\det(A + B) = \det(A) + \det(B)$.  
\[ \text{F} \]

(e) There exists a transition matrix where $-3$ is an eigenvalue.  
\[ \text{F} \]

(f) A transition matrix always has at least one eigenvalue.  
\[ \text{T} \]

(g) The formula $\langle x, y \rangle = 2x_1y_1$ defines an inner product on $\mathbb{R}^2$.  
\[ \text{F} \]

(h) The formula $\langle f, g \rangle = \int_0^1 f(t)g(t)dt$ defines an inner product on $P_2(\mathbb{R})$.  
\[ \text{T} \]

**Extra Credit** Can you find a $2 \times 2$ matrix $A$ such that $A^3 = A$ but $A$ has no eigenvalues. Explain.

**Solution:** We claim that there is no such matrix.

Note that $A$ must be invertible, otherwise 0 is an eigenvalue. Thus $A^2 = I$. The matrix

$$ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} $$

satisfies $A^2 = I$ implies either $b = c = 0, a^2 = d^2 = 1$ or $a = -d, bc = 1 - a^2$. In the first case the matrix is diagonal hence is diagonalizable, in the second case $\begin{pmatrix} a & b \\ (1-a^2)/b & -a \end{pmatrix}$ has eigenvalues $\pm 1$ as the characteristic polynomial is $t^2 - 1$.

or

$A^3 = A$ implies $A(A-I)(A+I) = A(A^2-I) = 0$. Thus at least one of the matrices $A, A-I, A+I$ is non-invertible. Which means $A$ must have at least one real eigenvalue in $\{0, 1, -1\}$. 
