There are 75 points possible on this exam, spread out over 11 problems. Take care to note that the problems are not weighted equally. Calculators, phones, books, notes (beyond the allowed two pages) and suchlike aids to gracious living are not permitted. Show all your work as credit will not be given for correct answers without proper justification.

Do not start until instructed.

Good luck!
1. Let \( W = \{(x, y) \in \mathbb{R}^2 \mid y \geq 0\} \).

(a) Is \( W \) a subspace of \( \mathbb{R}^2 \)? Prove your answer. \( \text{(3 points)} \)

(b) If \( W \) is a subspace, find a basis for \( W \); otherwise, find a basis for \( \text{span}(W) \). \( \text{(2 points)} \)

2. Consider the vectors \( u_1 = (1, 0, 1) \), \( u_2 = (0, 1, 1) \), and \( u_3 = (1, 1, 2) \) in \( \mathbb{R}^3 \). Find a basis for \( U = \text{span}\{u_1, u_2, u_3\} \). Be sure to justify your answer. \( \text{(5 points)} \)
3. Suppose $v_1, v_2,$ and $v_3$ are nonzero vectors in a vector space $V$, and set $U = \text{span}(\{v_1, v_2\})$ and $W = \text{span}(\{v_1, v_2, v_3\})$. When $v_3 \notin U$, prove $\dim(W)$ is either 2 or 3. \hspace{1cm} (5 \text{ points})

4. Suppose $f(t)$ is the characteristic polynomial of $A \in M_{n \times n}(\mathbb{R})$. Prove that $f(0) = \det(A)$. \hspace{1cm} (3 \text{ points})
5. Consider the linear transformation $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ that acts as shown at right:

(a) Find the eigenvalues and eigenvectors of $T$ via a geometric argument from the picture. Be sure to explain your reasoning.

Hint: One of the eigenvectors is $(0, 1)$.  \(3 \text{ points}\)

(b) Find the matrix $[T]_\beta$ where $\beta = \{e_1, e_2\}$ is the standard basis for $\mathbb{R}^2$. Hint: There are several approaches here, but one is to first compute $[T]_\gamma$ where $\gamma$ a basis consisting of eigenvalues.

\(4 \text{ points}\)

(c) What is the rank of $T$?  \(1 \text{ point}\)

(d) What is the nullity of $T$?  \(1 \text{ point}\)
6. Consider the Markov chain with transition matrix $A = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$.

(a) Show that this Markov chain is regular. (2 points)

(b) Suppose the Markov chain is initially in the state corresponding to the second column. Find the limiting probability distribution of the Markov chain as the number of steps goes to infinity. 

**Hint:** Use that the Markov chain is regular to avoid diagonalizing $A$, which is a real pain as some of the eigenvalues are not real. (5 points)

(c) Is $A^t$ also the transition matrix of a Markov chain? (1 point)
7. Consider the linear transformation $T: M_{2 \times 2}(\mathbb{R}) \to P_2(\mathbb{R})$ that sends the matrix \[
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}
\] to the polynomial $a + b + (b + 2c)x + (c + 3d)x^2$.

(a) Consider the bases $\beta = \{E_{11}, E_{12}, E_{21}, E_{22}\}$ for $M_{2 \times 2}(\mathbb{R})$ and $\{1, x, x^2\}$ for $P_2(\mathbb{R})$; here, the matrix $E_{ij}$ is zero except in the $(i, j)$-entry where it is one. Compute the matrix $[T]_{\beta}^{\gamma}$.

(b) Consider the alternate basis $\delta = \{x^2, 2x, 1\}$ for $P_2(\mathbb{R})$. Use your answer in (a) and the appropriate change of basis matrix to compute $[T]_{\beta}^{\delta}$.

(c) Is $T$ an isomorphism? Prove your answer.
8. Suppose $V$ is $\mathbb{C}^n$, viewed as a vector space over $\mathbb{C}$, with its standard inner product. Suppose $T$ is a linear operator on $V$ which is an isometry (also called a unitary operator).

(a) Prove that $T$ has at least one eigenvector. \textbf{(3 points)}

(b) Prove that if $\lambda \in \mathbb{C}$ is an eigenvalue of $T$ then $|\lambda| = 1$. \textbf{(4 points)}

9. Suppose $W$ is a subspace of a finite-dimensional inner product space $V$. Let $T: V \to V$ be orthogonal projection onto the subspace $W$. Apply the Dimension Theorem to $T$ to give a proof that

\[ \dim(W) + \dim(W^\perp) = \dim(V) \]

\textbf{(5 points)}
10. Let $V$ be an inner product space with field of scalars $\mathbb{R}$. Suppose $S = \{u_1, u_2, u_3\}$ is an orthonormal subset of $V$.

(a) Let $U = \text{span}(S)$. Give a basis for $U$ and compute $\dim(U)$. (3 points)

(b) Prove that $S' = \left\{ w_1 = \frac{1}{\sqrt{2}}(u_1 + u_2), \ w_2 = \frac{1}{\sqrt{2}}(u_1 - u_2), \ w_3 = u_3 \right\}$ is also orthonormal. (3 points)

(c) Suppose $U$ is all of $V$. Find a basis for $\{w_1\}^\perp$. (2 points)
11. Consider $V = \mathbb{R}^2$ with the nonstandard inner product $\langle x, y \rangle = 4x_1y_1 + x_2y_2$.

(a) Prove directly from the axioms that the above formula defines an inner product on $V$. (4 points)

(b) Prove or disprove: the set $\beta = \{e_1, e_2\}$ is an orthonormal basis of $V$ with respect this nonstandard inner product. (1 point)

(c) Let $A = \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$. Consider the linear operators $L_A$ and $L_B$ on $V$, which again has the above nonstandard inner product. Prove that the adjoint of $L_A$ is $L_B$. Note: The formulas for $A$ and $B$ are correct as written. (5 points)