MATH 416: Abstract Linear Algebra
Midterm 3 – Spring 2022
DATE: April 20, 2022

FILL OUT THE INFORMATION IN THE BOX.

NAME: _____________________________  Netid: _______________

I pledge that the work on this exam is entirely my own.

Student signature: ____________________________________________

READ THE FOLLOWING INFORMATION.

• This is a 50-minute exam.
• There is one extra credit problem. The maximum score for this exam is 50.
• Books, notes, and other aids are not allowed except for one page of cheat sheet. Collaboration is forbidden.
• Show all steps to earn full credit. Multiple answers for any problem earn ZERO credit.
• Do not unstaple pages. Loose pages will be ignored.

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— Best of Luck! —
Q1. Let $A = \begin{pmatrix} 3 & -1 \\ 2 & 0 \end{pmatrix}$.

(a) Compute the characteristic polynomial of $A$. \hspace{1cm} \frac{\hspace{1cm}}{2}

(b) Find all eigenvalues of $A$. \hspace{1cm} \frac{\hspace{1cm}}{2}

(c) For each eigenvalue compute the corresponding eigenspace. \hspace{1cm} \frac{\hspace{1cm}}{2}

(d) Diagonalize $A$, giving a diagonal matrix $D$ and an invertible matrix $Q$ so that $D = Q^{-1}AQ$. \hspace{1cm} \frac{\hspace{1cm}}{3}

(e) At left is our usual visualization of $L_A: \mathbb{R}^2 \rightarrow \mathbb{R}^2$. Use your answer in (d) to draw a more informative picture of $L_A$ at right. \hspace{1cm} \frac{\hspace{1cm}}{1}
Q2. (a) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation such that

$$T(1, 0, 0) = (1, 2, 1), \quad T(0, 1, 0) = (0, -1, 2), \quad T(0, 0, 1) = (2, 2, 0).$$

Let $S$ be the unit sphere $\{(x, y, z) \mid x^2 + y^2 + z^2 \leq 1\}$ with volume $\frac{4\pi}{3}$. Find the volume of $T(S)$.

(b) Let $A, B \in \mathcal{M}_{3 \times 3}(\mathbb{R})$ such that

$$AB = \begin{pmatrix} 3 & 1 & 2 \\ 2 & -2 & 2 \\ 0 & 1 & -1 \end{pmatrix}.$$

Find $\det(BA^2B)$. Justify each step.

(c) The constant term of a polynomial $f(t) = a_n t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + a_0$ is the final coefficient $a_0$. For the characteristic polynomial of an $n \times n$ matrix $A$, prove that the constant term is $\det(A)$. 

\[\text{---/3}\]
(This page intentionally left blank. You can use it for scratch work.)
Q3. Suppose $A \in \mathcal{M}_{3 \times 3}(\mathbb{R})$ is a transition matrix associated to a Markov chain.

(a) If $1/2$ and $1/3$ are eigenvalues of $A$, **prove** that $A$ is **diagonalizable**. $\quad$ ___/4

(b) **Give a transition matrix** $B$ where $1/2$ and $1/3$ are eigenvalues and justify your answer. $\quad$ ___/3

(c) If $\mathcal{N}(A - I) = \{ (t, 3t, 4t) \mid t \in \mathbb{R} \}$, **compute** (with justification) the matrix $\quad$ ___/3

\[
\lim_{n \to \infty} A^n = \begin{pmatrix}
& & \\
& & \\
& & 
\end{pmatrix}
\]
(This page intentionally left blank. You can use it for scratch work.)
Q4. (a) Let $T : \mathcal{M}_{2 \times 2}(\mathbb{R}) \to \mathcal{M}_{2 \times 2}(\mathbb{R})$ be the $T(A) = A^t$. Compute the characteristic polynomial of $T$ and find the eigenvalues of $T$. Hint: Use the standard basis.

(b) Let $U : V \to V$ be a linear transformation such that $U^2 = U$ and $U \neq I_V$. Prove that $0$ is an eigenvalue of $U$.

(c) Suppose the only eigenvalues of $B \in \mathcal{M}_{4 \times 4}(\mathbb{R})$ are $3, 4$ with $\dim(E_3) = \dim(E_4) = 2$. Determine whether $B$ is diagonalizable. Explain your answer.
Q5. Circle true or false as appropriate; you **DO NOT** need to provide any justification.

(a) The matrix \( A = \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix} \) can be written as a product of elementary matrices.

   TRUE    FALSE

(b) If a \( n \times n \) matrix has \( n \) distinct real eigenvalues, it is diagonalizable.

   TRUE    FALSE

(c) If \( A \) and \( B \) are \( n \times n \) matrices, then \( \det(AB) = \det(BA) \).

   TRUE    FALSE

(d) If all eigenvalues of \( A \in \mathcal{M}_{n\times n}(\mathbb{R}) \) are 1 then \( A = I_n \).

   TRUE    FALSE

(e) For any eigenvalues geometric multiplicity is always bigger than or equal to algebraic multiplicity.

   TRUE    FALSE

(f) If \( A \) and \( B \) are similar matrices, then \( A^3 \) and \( B^3 \) are also similar.

   TRUE    FALSE

(g) There exists a transition matrix where \(-1\) is an eigenvalue.

   TRUE    FALSE

(h) A transition matrix is always diagonalizable.

   TRUE    FALSE

(i) The vectors \( \{(1, 0), (i, 0), (0, 1), (0, i)\} \) are a basis for \( \mathbb{C}^2 \) regarded as a vector space over \( \mathbb{R} \).

   TRUE    FALSE

(j) The formula \( \langle f, g \rangle = \int_0^1 f(t)g(t) \, dt \) defines an inner product on \( \mathcal{P}_2(\mathbb{C}) \).

   TRUE    FALSE
(This page intentionally left blank. You can use it for scratch work.)
Extra Credit  Suppose $A \in M_{n \times n}(\mathbb{R})$ satisfies $A^2 = 3A$. Show that $A$ has at least one real eigenvalue.
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