Homework 8a

MATH 416: Abstract Linear Algebra

Due date: Not graded

1. In $C([0,1])$, let $f(t) = t$ and $g(t) = e^t$. Compute $\langle f, g \rangle = \int_0^1 f(t)g(t)dt, \|f\|, \|g\|, \text{and} \|f + g\|$. Then verify both the Cauchy–Schwarz inequality and the triangle inequality.

Solution: As given in the definition, we compute

$$\langle f, g \rangle = \int_0^1 f(t)g(t)dt = \int_0^1 te^t dt = \left. (te^t - e^t) \right|_0^1 = 1.$$

Similarly, we compute $\|f\|^2 = \frac{1}{3}, \|g\|^2 = \frac{1}{2}(e^2 - 1)$ and

$$\|f + g\|^2 = \|f\|^2 + 2\langle f, g \rangle + \|g\|^2 = \frac{1}{3} + 2 + \frac{1}{2}(e^2 - 1) = \frac{1}{6}(3e^2 + 11).$$

And the two inequalities hold since

$$1 \leq \sqrt[3]{\frac{1}{3}} \sqrt[2]{\frac{1}{2}(e^2 - 1)} \text{ and } \sqrt[6]{\frac{1}{3}} + \sqrt[2]{\frac{1}{2}(e^2 - 1)} \leq \sqrt[3]{\frac{1}{3}} \sqrt[2]{\frac{1}{2}(e^2 - 1)}.$$

2. Provide reasons why each of the following is not an inner product on the given vector spaces.

(a) $\langle (a, b), (c, d) \rangle = ac - bd$ on $\mathbb{R}^2$.

(b) $\langle A, B \rangle = \text{Tr}(A + B)$ on $M_{2 \times 2}(\mathbb{R})$.

Solution:

(a) The inner product of a nonzero vector $(1, 1)$ and itself is $1 - 1 = 0$.

(b) Let $A = B = I_2$. We have $\langle 2A, B \rangle = 3 \neq 2\langle A, B \rangle = 4$.

3. Let $\beta$ be a basis for a finite-dimensional inner product space.

(a) Prove that, if $\langle x, z \rangle = 0$ for all $z \in \beta$, then $x = 0$.

(b) Prove that if $\langle x, z \rangle = \langle y, z \rangle$ for all $z \in \beta$, then $x = y$.

Solution: (a) We can represent $x$ as a linear combination of vectors from $\beta$ as $x = \sum_{i=1}^{k} a_i z_i$ where $\beta = \{z_1, z_2, \ldots, z_k\}$. Then we have

$$\langle x, x \rangle = \langle x, \sum_{i=1}^{k} a_i z_i \rangle = \sum_{i=1}^{k} a_i \langle x, z_i \rangle = 0.$$

Thus $x = 0$.

(b) This means that $\langle x - y, z \rangle = 0$ for all $z \in \beta$. So we have $x - y = 0$ and $x = y$.

4. Let $V$ be an inner product space.

(a) Suppose that $x$ and $y$ are orthogonal vectors in $V$. Prove that $\|x + y\|^2 = \|x\|^2 + \|y\|^2$. Deduce the Pythagorean theorem in $\mathbb{R}^2$.

(b) Prove that, $|\langle x, y \rangle| = \|x\| \cdot \|y\|$ if and only if one of the vectors $x$ or $y$ is a multiple of the other.

Solution:
(a) Two vectors are orthogonal means that the inner product of them is 0. So we have

\[ \|x + y\|^2 = \langle x + y, x + y \rangle = \|x\|^2 + \langle x, y \rangle + \langle y, x \rangle + \|y\|^2 = \|x\|^2 + \|y\|^2. \]

To deduce the Pythagorean Theorem in \( \mathbb{R}^2 \), we may begin by a right triangle \( ABC \) with the right angle \( B \). Assume that \( x = AB \) and \( y = BC \). Thus we know the length of two leg is \( \|x\| \) and \( \|y\| \). Finally we know \( AC = x + y \) and so the length of the hypotenuse is \( \|x + y\| \). Apply the proven equality and get the desired result.

(b) If one of \( x \) or \( y \) is zero, then the equality holds naturally and we have \( y = 0x \) or \( x = 0y \). So we may assume that both \( x, y \) are non-zero. Now if \( x = cy \), we have \( |\langle x, y \rangle| = |\langle cy, y \rangle| = |c| \|y\|^2 \) and \( \|x\| \cdot \|y\| = \|cy\| \cdot \|y\| = |c| \|y\|^2 \). For the necessity, we observe that if the equality holds, then we have \( \|x - cy\| = 0 \) where \( c = \langle x, y \rangle / \|x\| \|y\| \). So \( x = cy \).