Homework 8a

MATH 416: Abstract Linear Algebra

Due date: Not graded

1. In $C([0, 1])$, let $f(t) = t$ and $g(t) = e^t$. Compute $\langle f, g \rangle = \int_0^1 f(t)g(t)dt$, $\|f\|$, $\|g\|$, and $\|f + g\|$. Then verify both the Cauchy–Schwarz inequality and the triangle inequality.

2. Provide reasons why each of the following is not an inner product on the given vector spaces.
   (a) $\langle (a, b), (c, d) \rangle = ac - bd$ on $\mathbb{R}^2$.
   (b) $\langle A, B \rangle = \text{Tr}(A + B)$ on $\mathcal{M}_{2 \times 2}(\mathbb{R})$.

3. Let $\beta$ be a basis for a finite-dimensional inner product space.
   (a) Prove that, if $\langle x, z \rangle = 0$ for all $z \in \beta$, then $x = 0$.
   (b) Prove that if $\langle x, z \rangle = \langle y, z \rangle$ for all $z \in \beta$, then $x = y$.

4. Let $V$ be an inner product space.
   (a) Suppose that $x$ and $y$ are orthogonal vectors in $V$. Prove that $\|x + y\|^2 = \|x\|^2 + \|y\|^2$. Deduce the Pythagorean theorem in $\mathbb{R}^2$.
   (b) Prove that, $|\langle x, y \rangle| = \|x\| \cdot \|y\|$ if and only if one of the vectors $x$ or $y$ is a multiple of the other.