Homework 7

MATH 416: Abstract Linear Algebra

Due date: April 6, 2022

Each problem is worth 10 points and only five randomly chosen problems will be graded. Please indicate whom you worked with, it will not affect your grade in any way.

1. Let $M$ be a square matrix of the form 
   \[
   \begin{pmatrix}
   A & B \\
   0 & I_n
   \end{pmatrix}
   \]
   for some matrix $A \in \mathcal{M}_{k \times k}(\mathbb{R})$, $B \in \mathcal{M}_{k \times n}$ and the identity matrix $I_n$. Using induction on $n$, prove that 
   \[\det(M) = \det(A).\]

   **Solution:** The matrix $A$ must be a square matrix, for $M$ to be a square matrix. Let $k$ be the number of rows of $A$.
   The statement is true for $n = 1$, by expanding along the last row with only one entry being 1. Suppose it is true for $n - 1$. For $n$, expanding along the last row with only one entry being 1, we have 
   \[
   \det\begin{pmatrix}
   A & B \\
   0 & I_n
   \end{pmatrix} = (-1)^{k+n+k+n} \cdot 1 \cdot \det\begin{pmatrix}
   A & B' \\
   0 & I_{n-1}
   \end{pmatrix} = \det(A)
   \]
   where $B'$ is the matrix obtained from $B$ by deleting the last column and the second equality follows by induction hypothesis.

2. For each of the following linear operators $T$ on a vector space $V$ and ordered bases $\beta$, compute $[T]_\beta$, and determine whether $\beta$ is a basis consisting of eigenvectors of $T$.
   (a) $V = \mathbb{R}^2$, $T(a, b) = (10a - 6b, 17a - 10b)$ and $\beta = \{(1, 2), (2, 3)\}$.
   (b) $V = \mathbb{R}^3$, $T(a, b, c) = (3a + 2b - 2c, -4a - 3b + 2c, -c)$ and $\beta = \{(0, 1, 1), (1, -1, 0), (1, 0, 2)\}$.

   **Solution:**
   (a) No. We have $[T]_\beta = \begin{pmatrix} 0 & 2 \\ -1 & 0 \end{pmatrix}$.
   (b) Yes. We have $[T]_\beta = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$.

3. Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} \in \mathcal{M}_{2 \times 2}(\mathbb{R})$.
   (a) Determine all the eigenvalues of $A$.
   (b) For each eigenvalue $\lambda$ of $A$, find the set of eigenvectors corresponding to $\lambda$.
   (c) If possible, find a basis for $\mathbb{R}^2$ consisting of eigenvectors of $A$.
   (d) If successful in finding such a basis, determine an invertible matrix $Q$ and a diagonal matrix $D$ such that $Q^{-1}AQ = D$.

   **Solution:** The characteristic polynomial for $A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$ is $t^2 - 3t - 4 = (t - 4)(t + 1)$ and the zeroes of it are 4 and $-1$. For eigenvalue 4, we may find $\mathcal{N}(A - 4I)$ and choose any basis, say $(2, 3)$. Similarly we can find the eigenvector corresponding to $-1$ is $(1, -1)$. Take 
   $\gamma = \{(2, 3), (1, -1)\}$
and
\[ Q = [I_2]_\gamma = \begin{pmatrix} 2 & 1 \\ 3 & -1 \end{pmatrix} \]

where \( \beta \) is the standard basis for \( \mathbb{R}^2 \). Then we know that
\[ Q^{-1}A = \begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix} \]

4. For each linear operator \( T \) on \( V \), find the eigenvalues of \( T \) and an ordered basis \( \beta \) for \( V \) such that \( [T]_\beta \) is a diagonal matrix.

(a) \( V = \mathbb{R}^3 \) and \( T(a, b, c) = (7a - 4b + 10c, 4a - 3b + 8c, -2a + b - 2c) \).

(b) \( V = M_{2\times2}(\mathbb{R}) \) and \( T \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & b \\ c & a \end{pmatrix} \).

Solution:

(a) \( \beta = \{ (2, 0, -1), (1, 2, 0), (1, -1, -1) \} \) and \( [T]_\beta = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \).

(b) \( \beta = \{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \} \) and \( [T]_\beta = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \).

5. Let \( T \) be a linear operator on a finite-dimensional vector space \( V \).

(a) Show that \( T \) is invertible if and only if \( 0 \) is not an eigenvalue of \( T \).

(b) If \( T \) is invertible, show that \( \lambda^{-1} \) is an eigenvalue of \( T^{-1} \) if and only if \( \lambda \) is an eigenvalue of \( T \).

Solution:

(a) \( T \) is invertible if and only if \( \mathcal{N}(T - 0 \cdot I_V) = \{ 0 \} \) if \( 0 \) is not an eigenvalue of \( T \).

(b) If \( T \) is invertible, all eigenvalues of \( T \) and \( T^{-1} \) are non-zero, by part (a). For \( \lambda \neq 0 \), note that \( T^{-1}v = \lambda^{-1}v \) iff \( Tv = \lambda T(\lambda^{-1}v) = \lambda(T^{-1}v) = \lambda v \). Thus, \( \lambda^{-1} \) is an eigenvalue of \( T^{-1} \) iff there is a non-zero vector \( v \in V \) such that \( T^{-1}v = \lambda^{-1}v \) iff there is a non-zero vector \( v \in V \) such that \( Tv = \lambda v \) iff \( \lambda \) is an eigenvalue of \( T \).

6. Suppose \( T : V \to V \) is a linear operator on a finite-dimensional vector space \( V \). Suppose \( v \in V \) is an eigenvector of \( T \) with eigenvalue \( \lambda \). As usual, \( T^m : V \to V \) denotes composition of \( T \) with itself \( m \) times. Prove that \( v \) is also an eigenvector for \( T^m \) and give a formula for the corresponding eigenvalue.

Solution: We have \( Tv = \lambda v \). Applying \( T \) on both sides we get \( T^2v = T(Tv) = T(\lambda v) = \lambda T(Tv) = \lambda^2 v \). By induction, we get \( T^m v = \lambda^m v \) for all \( m \geq 1 \). Thus \( v \) is an eigenvector of \( T^m \) with corresponding eigenvalue \( \lambda^m \).