Homework 4

MATH 416: Abstract Linear Algebra

Due date: March 2, 2022

Each problem is worth 10 points and only five randomly chosen problems will be graded. Please indicate whom you worked with, it will not affect your grade in any way.

1. For the given functions, prove that $T$ is a linear transformation, and find bases for both $\mathcal{N}(T)$ and $\mathcal{R}(T)$. Then compute the nullity and rank of $T$, and verify the dimension theorem. Finally, use the appropriate theorems in this section to determine whether $T$ is one-to-one or onto.

(a) $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(a_1, a_2, a_3) = (a_1 - a_2, 2a_3)$.

(b) $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(a_1, a_2) = (a_1 + a_2, 0, 2a_1 - a_2)$.

(c) $T : \mathcal{P}_2(\mathbb{R}) \to \mathcal{P}_3(\mathbb{R})$ defined by $T(f(x)) = xf(x) + f'(x)$.

**Solution:**

(a) It is a linear transformation since we have

$$T((a_1, a_2, a_3) + (b_1, b_2, b_3)) = T(a_1 + b_1, a_2 + b_2, a_3 + b_3)$$

$$= (a_1 + b_1 - a_2 - b_2, 2a_3 + 2b_3)$$

$$= (a_1 - a_2, 2a_3) + (b_1 - b_2, 2b_3) = T(a_1, a_2, a_3) + T(b_1, b_2, b_3)$$

and

$$T(c(a_1, a_2, a_3)) = (c(a_1 - a_2), 2ca_3) = cT(a_1, a_2, a_3).$$

We have $\mathcal{N}(T) = \{(a, a, 0) \mid a \in \mathbb{R}\}$ with basis $\{(1, 1, 0)\}$; $\mathcal{R}(T) = \mathbb{R}^2$ with basis $\{(1, 0, 0), (0, 1, 0)\}$. Hence $T$ is not one-to-one but onto.

(b) Similar to (a), check that $T$ is a linear transformation. $\mathcal{N}(T) = \{0\}$ with basis $\emptyset$; $\mathcal{R}(T) = \{a_1(1, 0, 2) + a_2(1, 0, -1) \mid a_1, a_2 \in \mathbb{R}\}$ with basis $\{(1, 0, 2), (1, 0, -1)\}$. Hence $T$ is one-to-one but not onto.

(c) Here we have $T(a + bx + cx^2) = b + (a + 2c)x + bx^2 + cx^3$. Check that this is also a linear transformation with $\mathcal{N}(T) = \{0\}$ with basis $\emptyset$; $\mathcal{R}(T) = \{a + bx + ax^2 + cx^3 \mid a, b, c \in \mathbb{R}\}$ with basis $\{1 + x^2, x, x^3\}$. Hence, $T$ is one-to-one but not onto.

2. (a) Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^2$ is linear, $T(1, 0) = (1, 4)$ and $T(1, 1) = (2, 5)$. What is $T(2, 3)$? Is $T$ one-to-one?

(b) Give an example of a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\mathcal{N}(T) = \mathcal{R}(T)$.

**Solution:**

(a) We may take $U(a, b) = a(1, 4) + b(1, 1)$. By the fact that $\{(1, 0), (1, 1)\}$ is a basis for $\mathbb{R}^2$ and a linear transformation is uniquely determined by its value on a basis, we must have $T = U$. Hence we have $T(2, 3) = (5, 11)$. Also, $T$ is one-to-one as $(1, 4), (2, 5)$ are linearly independent and thus $\mathcal{N}(T) = \{(0, 0)\}$.

(b) Let $T(x, y) = (y, 0)$. Then we have $\mathcal{N}(T) = \mathcal{R}(T) = \{(x, 0) \mid x \in \mathbb{R}\}$.

3. Let $V, W$ be vector spaces, with $\dim(V) = n, \dim(W) = m$, and $n > m$.

(a) Show that there is no one-to-one linear transformation $T : V \to W$.

(b) Show that there is no onto linear transformation $T : W \to V$ (notice that $V, W$ have flipped in this expression!)

(c) Show that a linear map $T : V \to W$ need not be onto by giving an example where it is not.

**Solution:**

(a) Because $\text{nullity}(T) = \dim(V) - \text{rank}(T) \geq \dim(V) - \dim(W) = n - m > 0$ by Dimension Theorem, we have $\mathcal{N}(T) \neq \{0\}$.

(b) Because $\text{rank}(T) \leq \dim(W) = m < n = \dim(V)$ by Dimension Theorem, we have $\mathcal{R}(T) \neq V$. 

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(c) Take $V = \mathbb{R}^3$, $W = \mathbb{R}^2$ and the linear transformation $T : V \rightarrow W$ given by $T(x,y,z) = (x,x)$. Clearly, $\mathcal{N}(T) = \{(0,y,z) \mid y,z \in \mathbb{R}\}$ and $\mathcal{R}(T) = \{(x,x) \mid x \in \mathbb{R}\}$. Hence $T$ is not one-one and not onto.

4. Given bases $\beta$ and $\gamma$ of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, for each linear transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$, compute $[T]_\beta^\gamma$.

(a) $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ with $\beta, \gamma$ standard bases and $T(a_1, a_2) = (2a_1 - a_2, 3a_1 + 4a_2, a_1)$.

(b) $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ with $\beta, \gamma$ standard bases and $T(a_1, a_2, a_3) = (2a_1 + 3a_2 - a_3, a_1 + a_3)$.

(c) $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ with $\beta$ standard basis for $\mathbb{R}^2$, $\gamma = \{(1,1,0),(0,1,1),(2,2,3)\}$ and $T(a_1, a_2) = (a_1 - a_2, a_1, 2a_1 + a_2)$.

(d) $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ with $\beta = \{(1,2),(2,3)\}$, $\gamma = \{(1,1,0),(0,1,1),(2,2,3)\}$ and $T(a_1, a_2) = (a_1 - a_2, a_1, 2a_1 + a_2)$.

**Solution:** (a) We have $T(1,0) = (2,3,1) = 2(1,0,0) + 3(0,1,0) + 1(0,0,1)$ and $T(0,1) = (-1,4,0) = -1(1,0,0) + 4(0,1,0) + 0(0,0,1)$. Hence we get

$$[T]_\beta^\gamma = \begin{pmatrix} 2 & -1 \\ 3 & 4 \\ 1 & 0 \end{pmatrix}.$$  

(b) $[T]_\beta^\gamma = \begin{pmatrix} 2 & 3 & -1 \\ 1 & 0 & 1 \end{pmatrix}$.

(c) Since

$$T(1, 0) = (1,1,2) = \frac{-1}{3}(1,1,0) + 0(0,1,1) + \frac{2}{3}(2,2,3),$$

$$T(0, 1) = (-1,0,1) = -1(1,1,0) + 1(0,1,1) + 0(2,2,3),$$

we have

$$[T]_\beta^\gamma = \begin{pmatrix} -\frac{1}{3} & -1 \\ 0 & 1 \\ \frac{2}{3} & 0 \end{pmatrix}.$$  

(d) Since

$$T(1, 2) = (-1,1,4) = \frac{-7}{3}(1,1,0) + 2(0,1,1) + \frac{2}{3}(2,2,3),$$

$$T(2, 3) = (-1,2,7) = -\frac{11}{3}(1,1,0) + 3(0,1,1) + \frac{4}{3}(2,2,3),$$

we have

$$[T]_\beta^\gamma = \begin{pmatrix} -\frac{7}{3} & -\frac{11}{3} \\ 2 & \frac{3}{3} \\ \frac{2}{3} & 4 \end{pmatrix}.$$  

5. Let

$$\alpha = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}, \quad \beta = \{1, x, x^2\} \quad \text{and} \quad \gamma = \{1\}.$$

(a) Define $T : \mathcal{M}_{2 \times 2}(\mathbb{R}) \rightarrow \mathcal{M}_{2 \times 2}(\mathbb{R})$ by $T(A) = A^t$. Compute $[T]_\alpha$.

(b) Define $T : \mathcal{P}_2(\mathbb{R}) \rightarrow \mathcal{M}_{2 \times 2}(\mathbb{R})$ by $T(f) = \begin{pmatrix} f'(0) \\ 0 \\ f''(3) \end{pmatrix}$, where $'$ denotes differentiation. Compute $[T]_\beta^\gamma$.

(c) Define $T : \mathcal{M}_{2 \times 2}(\mathbb{R}) \rightarrow \mathbb{R}$ by $T(A) = \text{Tr}(A) = \text{sum of diagonal elements of } A$. Compute $[T]_\alpha^\gamma$.

(d) Define $T : \mathcal{P}_2(\mathbb{R}) \rightarrow \mathbb{R}$ by $T(f(x)) = f(2)$. Compute $[T]_\beta^\gamma$.

(e) If $A = \begin{pmatrix} 1 & -2 \\ 0 & 4 \end{pmatrix}$ compute $[A]_\alpha$.

(f) If $f(x) = 3 - 6x + x^2$, compute $[f]_\beta$. 


For \( a \in \mathbb{R} \), compute \([a]_{\gamma}\).

**Solution:**

(a) \([T]_{\alpha} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\).

(b) \([T]_{\beta}^{\alpha} = \begin{pmatrix} 0 & 1 & 0 \\ 2 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}\).

(c) \([T]_{\alpha}^{\gamma} = \begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix}\).

(d) \([T]_{\beta}^{\gamma} = \begin{pmatrix} 1 & 2 & 4 \end{pmatrix}\).

(e) \([A]_{\alpha} = \begin{pmatrix} 1 \\ -2 \\ 0 \\ 4 \end{pmatrix}\).

(f) \([f(x)]_{\beta} = \begin{pmatrix} 3 \\ -6 \\ 1 \end{pmatrix}\).

(g) \([a]_{\gamma} = (a)\).

6. We define the linear transformation \( T_\theta : \mathbb{R}^2 \to \mathbb{R}^2 \) to be rotation counter-clockwise about the origin through angle \( \theta \). Let \( T_x \) be the transformation that reflects in the x-axis.

(a) Write down the matrices of \( T_\theta \) and \( T_x \) with the respect to the standard basis \( \beta = \{(1, 0), (0, 1)\} \) for \( \mathbb{R}^2 \).

(b) Show that for \( \theta \in (0, \pi) \cup (\pi, 2\pi) \) one has \( T_x \circ T_\theta \neq T_\theta \circ T_x \).

(c) Next, show that there is some angle \( \psi \) such that \( T_x \circ T_\psi = T_\theta \circ T_x \).

What is the relationship between \( \theta \) and \( \psi \)? Discuss the geometric meaning of this computation.

**Solution:**

(a) We have \( T_\theta(1, 0) = (\cos \theta, \sin \theta), T_\theta(0, 1) = (-\sin \theta, \cos \theta) \). Thus we have

\[
[T_\theta(1, 0)]_{\beta} = \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}, \quad [T_\theta(0, 1)]_{\beta} = \begin{pmatrix} -\sin \theta \\ \cos \theta \end{pmatrix}, \quad \text{and} \quad [T_\theta]_{\beta} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}.
\]

(b) Note that,

\[
[T_x \circ T_\theta]_{\beta} = [T_x]_{\beta}[T_\theta]_{\beta} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta \\ -\sin \theta & -\cos \theta \end{pmatrix}.
\]

\[
[T_\theta \circ T_x]_{\beta} = [T_\theta]_{\beta}[T_x]_{\beta} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix}.
\]

For \( \theta \in (0, \pi) \cup (\pi, 2\pi) \), we have \( \sin \theta \neq -\sin \theta \) and thus

\( T_x \circ T_\theta \neq T_\theta \circ T_x \).

(c) Taking \( \psi = -\theta \) we get

\[
[T_x \circ T_\psi]_{\beta} = \begin{pmatrix} \cos \psi & -\sin \psi \\ -\sin \psi & -\cos \psi \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix}.
\]

\[
[T_\theta \circ T_x]_{\beta} = \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix}.
\]
Thus

$$T_x \circ T_{-\theta} = T_\theta \circ T_x.$$

Geometrically, this means that first reflecting in the $x$-axis and then rotating counter-clockwise by angle $\theta$ is same as first rotating clockwise by angle $\theta$ and then reflecting in the $x$-axis.