Homework 4

MATH 416: Abstract Linear Algebra

Due date: March 2, 2022

Each problem is worth 10 points and only five randomly chosen problems will be graded. Please indicate whom you worked with, it will not affect your grade in any way.

1. For the given functions, prove that $T$ is a linear transformation, and find bases for both $\mathcal{N}(T)$ and $\mathcal{R}(T)$. Then compute the nullity and rank of $T$, and verify the dimension theorem. Finally, use the appropriate theorems in this section to determine whether $T$ is one-to-one or onto.

(a) $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(a_1,a_2,a_3) = (a_1-a_2,2a_3)$.

(b) $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(a_1,a_2) = (a_1+a_2,0,2a_1-a_2)$.

(c) $T : \mathcal{P}_2(\mathbb{R}) \to \mathcal{P}_3(\mathbb{R})$ defined by $T(f(x)) = xf(x) + f'(x)$.

2. (a) Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^2$ is linear, $T(1,0) = (1,4)$ and $T(1,1) = (2,5)$. What is $T(2,3)$? Is $T$ one-to-one?

(b) Give an example of a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\mathcal{N}(T) = \mathcal{R}(T)$.

3. Let $V, W$ be vector spaces, with $\dim(V) = n, \dim(W) = m$, and $n > m$.

(a) Show that there is no one-to-one linear transformation $T : V \to W$.

(b) Show that there is no onto linear transformation $T : W \to V$ (notice that $V, W$ have flipped in this expression!)

(c) Show that a linear map $T : V \to W$ need not be onto by giving an example where it is not.

4. Given bases $\beta$ and $\gamma$ of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, for each linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$, compute $[T]_\beta^\gamma$.

(a) $T : \mathbb{R}^2 \to \mathbb{R}^3$ with $\beta, \gamma$ standard bases and $T(a_1,a_2) = (2a_1-a_2,3a_1+4a_2,a_1)$.

(b) $T : \mathbb{R}^3 \to \mathbb{R}^2$ with $\beta, \gamma$ standard bases and $T(a_1,a_2,a_3) = (2a_1+3a_2-a_3,a_1+a_3)$.

(c) $T : \mathbb{R}^2 \to \mathbb{R}^3$ with $\beta$ standard basis for $\mathbb{R}^2$, $\gamma = \{(1,1,0),(0,1,1),(2,2,3)\}$ and $T(a_1,a_2) = (a_1-a_2,a_1,2a_1+a_2)$.

(d) $T : \mathbb{R}^2 \to \mathbb{R}^3$ with $\beta = \{(1,2),(2,3)\}$, $\gamma = \{(1,1,0),(0,1,1),(2,2,3)\}$ and $T(a_1,a_2) = (a_1-a_2,a_1,2a_1+a_2)$.

5. Let

$$\alpha = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}, \quad \beta = \{1, x, x^2\} \quad \text{and} \quad \gamma = \{1\}.$$

(a) Define $T : \mathcal{M}_{2 \times 2}(\mathbb{R}) \to \mathcal{M}_{2 \times 2}(\mathbb{R})$ by $T(A) = A'$. Compute $[T]_\alpha^\gamma$.

(b) Define

$$T : \mathcal{P}_2(\mathbb{R}) \to \mathcal{M}_{2 \times 2}(\mathbb{R}) \quad \text{by} \quad T(f) = \begin{pmatrix} f'(0) & 2f(1) \\ 0 & f''(3) \end{pmatrix},$$

where $'$ denotes differentiation. Compute $[T]_\beta^\gamma$.

(c) Define $T : \mathcal{M}_{2 \times 2}(\mathbb{R}) \to \mathbb{R}$ by $T(A) = \text{Tr}(A) = \text{sum of diagonal elements of } A$. Compute $[T]_\alpha^\gamma$.

(d) Define $T : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$ by $T(f(x)) = f(2)$. Compute $[T]_\beta^\gamma$.

(e) If $A = \begin{pmatrix} 1 & -2 \\ 0 & 4 \end{pmatrix}$ compute $[A]_\alpha$.

(f) If $f(x) = 3 - 6x + x^2$, compute $[f]_\beta$. 

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(g) For $a \in \mathbb{R}$, compute $[a]_\gamma$.

6. We define the linear transformation $T_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ to be rotation counter-clockwise about the origin through angle $\theta$. Let $T_x$ be the transformation that reflects in the $x$-axis.

(a) Write down the matrices of $T_\theta$ and $T_x$ with respect to the standard basis $\beta = \{(1, 0), (0, 1)\}$ for $\mathbb{R}^2$.

(b) Show that for $\theta \in (0, \pi) \cup (\pi, 2\pi)$ one has

$$T_x \circ T_\theta \neq T_\theta \circ T_x.$$ 

(c) Next, show that there is some angle $\psi$ such that

$$T_x \circ T_\psi = T_\theta \circ T_x.$$ 

What is the relationship between $\theta$ and $\psi$? Discuss the geometric meaning of this computation.