Homework 2

MATH 416: Abstract Linear Algebra

Due date: February 9, 2022

Each problem is worth 10 points and only five randomly chosen problems will be graded. Please indicate whom you worked with, it will not affect your grade in any way.

1. Solve each of the following linear systems by writing down its augmented matrix, doing row operations to get a matrix in reduced row echelon form, and using that to find all of the solutions. You should label your row operations as in RREF of [B].

(a) \[
\begin{align*}
2x_1 + x_2 &= 0 \\
x_1 + x_2 &= 1 \\
3x_1 + 4x_2 &= 5 \\
3x_1 + 5x_2 &= 7
\end{align*}
\]

(b) \[
\begin{align*}
y_1 + 2y_2 - y_3 &= 1 \\
y_1 + y_2 + 2y_3 &= 0 \\
5y_1 + 8y_2 + y_3 &= 1
\end{align*}
\]

(c) \[
\begin{align*}
2x_1 + 4x_2 + 5x_3 + 7x_4 &= 18 \\
x_1 + 2x_2 + x_3 - x_4 &= 3 \\
4x_1 + 8x_2 + 7x_3 + 5x_4 &= 24
\end{align*}
\]

2. (a) Suppose \( A \) is an \( m \times n \) matrix with \( m < n \). Show that the null space \( \mathcal{N}(A) \) contains a nonzero vector by an argument involving the reduced row echelon form of \( A \).

(b) Use part (a) to prove that any \( j \) vectors in \( \mathbb{R}^k \) are linearly dependent if \( j > k \).

3. (a) Suppose \( S \) is a subset of a vector space \( V \). Show that if \( v \in V \) is contained in \( \text{span}(S) \), then \( \text{span}(S) = \text{span}(S \cup \{v\}) \).

(b) Consider \( V = \mathbb{R}^2 \) and \( S = \{(x, y) \mid x \geq 0 \text{ and } y \geq x\} \). Use part (a) to give a short proof that \( \text{span}(S) = \mathbb{R}^2 \) by showing that \( \text{span}(S) \) contains the vectors \((1, 0)\) and \((0, 1)\).

4. Let \( u \) and \( v \) be distinct vectors in a vector space \( V \). Show that \( \{u, v\} \) is linearly dependent if and only if one of \( u \) or \( v \) is a scalar multiple of the other.

5. Either prove or give a counterexample to the following statement: If \( v_1, v_2, v_3 \) are three vectors in \( \mathbb{R}^3 \) none of which is a scalar multiple of another, then they are linearly independent.

6. In the vector space \( \mathcal{F}(\mathbb{R}, \mathbb{R}) \), all functions from \( \mathbb{R} \) to \( \mathbb{R} \) consider the elements \( f(t) = \sin(t) \) and \( g(t) = \cos(t) \). Is the subset \( \{f, g\} \) linearly dependent or linearly independent? Prove your answer.