Homework 10

MATH 416: Abstract Linear Algebra

Due date: May 4, 2022

Each problem is worth 10 points. Please indicate whom you worked with, it will not affect your grade in any way.

1. For each linear operator $T$ on an inner product space $V$, determine whether $T$ is normal, self-adjoint, or neither. If possible, produce an orthonormal basis of eigenvectors of $T$ for $V$ and list the corresponding eigenvalues.

   (a) $V = \mathbb{R}^2$ and $T$ is defined by $T(a, b) = (2a - 2b, -2a + 5b)$.
   (b) $V = \mathbb{C}^2$ and $T$ is defined by $T(a, b) = (2a + ib, a + 2b)$.
   (c) $V = M_{2\times2}(\mathbb{R})$ and $T$ is defined by $T(A) = A^t$.

2. Let $T$ and $U$ be self-adjoint operator on an inner product space $V$. Prove that $TU$ is self-adjoint if and only if $TU = UT$.

3. Let $T$ be a normal operator on a finite-dimensional inner product space $V$.

   (a) Prove that $\mathcal{N}(T) = \mathcal{N}(T^*)$ and $\mathcal{R}(T) = \mathcal{R}(T^*)$.
   (b) Prove that the subspaces $\mathcal{N}(T)$ and $\mathcal{R}(T)$ are orthogonal. **Hint:** Use the last problem in HW9.
   (c) Give an example of a (non-normal) linear operator $S$ where $\mathcal{N}(S) \neq \mathcal{N}(S^*)$ and $\mathcal{R}(S) \neq \mathcal{R}(S^*)$.

4. A matrix $A \in M_{n \times n}(\mathbb{R})$ is **Gramian** if there is a $B \in M_{n \times n}(\mathbb{R})$ such that $A = B^tB$. Prove that $A$ is Gramian if and only if $A$ is symmetric and all of its eigenvalues are non-negative. **Hint:** For $(\Leftarrow)$, note that $A$ is diagonalizable via an orthonormal basis $\{u_1, \ldots, u_n\}$ where $u_i$ is an eigenvector of $A$ with eigenvalue $\lambda_i$. Consider the linear operator $T$ on $\mathbb{R}^n$ where $T(u_i) = \sqrt{\lambda_i}u_i$. Now take $B = [T]_{\text{std}}$ and check that $A = B^tB$.

5. Find an orthogonal matrix whose first row is $(\frac{1}{3}, \frac{2}{3}, \frac{2}{3})$.  