Homework 1

MATH 416: Abstract Linear Algebra

Due date: February 2, 2022

Each problem is worth 10 points and only five randomly chosen problems will be graded. Please indicate whom you worked with, it will not affect your grade in any way.

1. Determine whether the following sets are subspaces of $\mathbb{R}^3$ under the operations of addition and scalar multiplication defined on $\mathbb{R}^3$. Justify your answers.
   
   (a) $W_1 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 = 3a_2, a_3 = -a_2\}$
   
   (b) $W_2 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 = a_3 + 2\}$
   
   (c) $W_3 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : 2a_1 - 7a_2 + a_3 = 0\}$
   
   (d) $W_4 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 - 4a_2 - a_3 = 0\}$
   
   (e) $W_5 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 + 2a_2 - 3a_3 = 1\}$

   Describe $W_1 \cap W_3, W_1 \cap W_4$ and $W_3 \cap W_4$, and observe that each is a subspace of $\mathbb{R}^3$.

2. A square matrix $A$ is called upper triangular if all entries lying below the diagonal are 0, that is, $A_{ij} = 0$ whenever $i > j$. Show that the upper triangular matrices form a subspace of $\mathcal{M}_{n \times n}(\mathbb{R})$.

3. Solve the following systems of linear equations by adding and multiplying equations.

   (a) $\begin{align*}
   2x_1 - 2x_2 - 3x_3 &= -2 \\
   3x_1 - 3x_2 - 2x_3 + 5x_4 &= 7 \\
   x_1 - x_2 - 2x_3 - x_4 &= -3
   \end{align*}$

   (b) $\begin{align*}
   3x_1 - 7x_2 + 4x_3 &= 10 \\
   x_1 - 2x_2 + x_3 &= 3 \\
   2x_1 - x_2 - 2x_3 &= 6
   \end{align*}$

4. For a nonempty set $S$, we use $\mathcal{F}(S, \mathbb{R})$ to denote the set of all functions from $S$ to $\mathbb{R}$; as described in Example 3 on page 9 of [FIS], this is a vector space over $\mathbb{R}$. Fix a point $s_0$ in $S$ and consider the subset $W$ of $\mathcal{F}(S, \mathbb{R})$ consisting of all functions where $f(s_0) = 0$.

   (a) Show that $W$ is a subspace of $\mathcal{F}(S, \mathbb{R})$.

   (b) Consider instead the subset where $f(s_0) = 1$. Is this also a subspace? Justify your answer.

5. In these questions, determine whether the first vector can be expressed as a linear combination of the other two and write the coefficients if the answer is yes.

   (a) $(-2, 0, 3), (1, 3, 0), (2, 4, -1)$.

   (b) $x^3 - 3x + 5, x^3 + 2x^2 - x + 1, x^3 + 3x^2 - 1$.

6. Suppose that $A$, $B$ and $C$, are $m \times n$ matrices with real coefficients. Prove the following three facts from the definition of row equivalence.

   (a) $A$ is row equivalent to $A$.

   (b) If $A$ is row equivalent to $B$, then $B$ is row equivalent to $A$.

   (c) If $A$ is row equivalent to $B$, and $B$ is row equivalent to $C$, then $A$ is row equivalent to $C$.

**Note:** A relationship that satisfies these three properties is known as an equivalence relation; this is a formal way of saying that a relationship behaves like equality, without requiring the relationship to be as strict as equality itself.