Write down a rigorous proof for the following results.

1. Given a vector \( \mathbf{u} \) in a vector space \( V \) over \( \mathbb{R} \), show that there is a unique vector \( \mathbf{w} \) such that \( \mathbf{u} + \mathbf{w} = \mathbf{0} \).

   **Solution:** Given a vector \( \mathbf{u} \in V \), suppose there are two vectors \( \mathbf{w}, \mathbf{v} \) such that \( \mathbf{u} + \mathbf{w} = \mathbf{0}, \mathbf{u} + \mathbf{v} = \mathbf{0} \). It is enough to show that \( \mathbf{w} = \mathbf{v} \), which will imply that there exists a unique additive inverse. We have

   \[
   \mathbf{w} + \mathbf{u} = \mathbf{u} + \mathbf{w} = \mathbf{0} = \mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u},
   \]

   By cancellation law for vector addition, we have \( \mathbf{w} = \mathbf{v} \). This completes the proof.

2. Let \( V \) denote the set of all \( m \times n \) matrices with real entries with the matrix addition and scalar multiplication given by component-wise addition and multiplication. Prove that, \( V \) is a vector space over \( \mathbb{R} \).

   **Solution:** We have

   \[
   V = \left\{ A = ((a_{ij}))_{1 \leq i \leq m; 1 \leq j \leq n} \mid a_{ij} \in \mathbb{R} \text{ for } 1 \leq i \leq m; 1 \leq j \leq n \right\}
   \]

   with vector addition given by component-wise addition and scalar multiplication given by component-wise scalar multiplication. We check the eight axioms of vector space.

1. For all \( A = ((a_{ij}))_{1 \leq i \leq m; 1 \leq j \leq n}, B = ((b_{ij}))_{1 \leq i \leq m; 1 \leq j \leq n} \in V \), we have

   \[
   A + B = ((a_{ij}))_{1 \leq i \leq m; 1 \leq j \leq n} + ((b_{ij}))_{1 \leq i \leq m; 1 \leq j \leq n}
   \]

   \[
   = ((a_{ij} + b_{ij}))_{1 \leq i \leq m; 1 \leq j \leq n}
   \]

   \[
   = ((b_{ij} + a_{ij}))_{1 \leq i \leq m; 1 \leq j \leq n}
   \]

   \[
   = ((b_{ij}))_{1 \leq i \leq m; 1 \leq j \leq n} + ((a_{ij}))_{1 \leq i \leq m; 1 \leq j \leq n}
   \]

   \[
   = B + A.
   \]

2. Associativity follows from component-wise associativity.

3. The zero vector is the zero matrix \( \mathbf{0} = ((0))_{1 \leq i \leq m; 1 \leq j \leq n} \) with all entries equal to 0 such that for \( A = ((a_{ij}))_{1 \leq i \leq m; 1 \leq j \leq n} \in V \) we have \( A + \mathbf{0} = A \).

4. For \( A = ((a_{ij}))_{1 \leq i \leq m; 1 \leq j \leq n} \in V \), the matrix \( B = ((-a_{ij}))_{1 \leq i \leq m; 1 \leq j \leq n} \) satisfies \( A + B = \mathbf{0} \).

5. For \( A = ((a_{ij}))_{1 \leq i \leq m; 1 \leq j \leq n} \in V \), we have \( 1 \cdot A = 1 \cdot ((a_{ij}))_{1 \leq i \leq m; 1 \leq j \leq n} = ((1 \cdot a_{ij}))_{1 \leq i \leq m; 1 \leq j \leq n} = A \).

6. Associativity of scalar multiplication follows from component-wise associativity for scalar multiplication.

7. For \( c \in \mathbb{R}, A = ((a_{ij}))_{1 \leq i \leq m; 1 \leq j \leq n} \in V, B = ((b_{ij}))_{1 \leq i \leq m; 1 \leq j \leq n} \in V \) we have

   \[
   c \cdot (A + B) = c \cdot ((a_{ij} + b_{ij}))_{1 \leq i \leq m; 1 \leq j \leq n}
   \]

   \[
   = ((ca_{ij} + cb_{ij}))_{1 \leq i \leq m; 1 \leq j \leq n}
   \]

   \[
   = ((ca_{ij}))_{1 \leq i \leq m; 1 \leq j \leq n} + ((cb_{ij}))_{1 \leq i \leq m; 1 \leq j \leq n}
   \]

   \[
   = c \cdot A + c \cdot B.
   \]

8. For \( c, d \in \mathbb{R}, A = ((a_{ij}))_{1 \leq i \leq m; 1 \leq j \leq n} \in V \) we have

   \[
   (c + d) \cdot A = (c + d) \cdot ((a_{ij}))_{1 \leq i \leq m; 1 \leq j \leq n}
   \]

   \[
   = ((ca_{ij} + da_{ij}))_{1 \leq i \leq m; 1 \leq j \leq n}
   \]

   \[
   = ((ca_{ij}))_{1 \leq i \leq m; 1 \leq j \leq n} + ((da_{ij}))_{1 \leq i \leq m; 1 \leq j \leq n}
   \]

   Thus \( V \) is a vector space under matrix addition and component-wise scalar multiplication.
3. For any scalar \( a \) in \( \mathbb{R} \), show that

\[ a \cdot 0 = 0. \]

**Solution:** We take a scalar \( a \in \mathbb{R} \). We have

\[
a \cdot 0 + a \cdot 0 = a \cdot (0 + 0) = a \cdot 0 = a \cdot 0 + 0 = 0 + a \cdot 0,
\]

where we used vector space axiom 7), 3), 3), 1) in the successive equalities, respectively. Now, using cancellation law for vector addition, we have \( a \cdot 0 = 0 \). This completes the proof.