MATH 416: Abstract Linear Algebra
Take Home Final – Spring 2022
DATE: May 13, 2022

FILL OUT THE INFORMATION IN THE BOX.

Name: ___________________________ Netid: _____________

I pledge that the work on this exam is entirely my own.

Student signature: _____________________________________________

READ THE FOLLOWING INFORMATION.

• This is a Take Home exam. Write your own solution.
• There is one extra credit problem. The maximum score for this exam is 60.
• Collaboration is forbidden. You can only use and refer materials covered in the class.
• Show all steps to earn full credit. Multiple answers for any problem earn ZERO credit.

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— Best of Luck! —
Q1. [10 pts.] Answer the following questions.

(a) Find the rank of the matrix \( A = \begin{pmatrix} 0 & 2 & 1 \\ 1 & 3 & 1 \\ -1 & 1 & 1 \end{pmatrix} \). 

(b) Does there exist a vector \( b \in \mathbb{R}^3 \) such that the linear system \( LS(A, b) \) has a unique solution? Explain.

(c) Consider the linear operator \( T \) on \( P_3(\mathbb{R}) \) given by

\[
T(p) = p(x) - 2p'(x) + p''(x).
\]

Let \( \beta = \{u_1 = 1, u_2 = x - 2, u_3 = x^2 - 2x, u_4 = x^3\} \) be an ordered basis for \( P_3(\mathbb{R}) \). Compute \( [T]_\beta \).

(d) Is \( T \) from (c) invertible? Justify your answer.

Q2. [10 pts.] Consider the vector space

\[
W = \text{Span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} \right\}.
\]

(a) Find a basis for \( W \).

(b) What is the dimension of \( W \)?

(c) Write down \( W^\perp \) as a standard subspace (null/col/row space of some matrix).

(d) Find a basis for \( W^\perp \).

Q3. [10 pts.] Consider the Markov chain with transition matrix \( P = \begin{pmatrix} 1/2 & 0 & 1/2 \\ 0 & 2/3 & 1/3 \\ 1/2 & 1/3 & 1/6 \end{pmatrix} \).

(a) Show that the Markov chain is regular.

(b) Is \( P^t \) also the transition matrix of a Markov chain?

(c) Without computation find an eigenvector of \( P \) with eigenvalue 1. Explain.

(d) Suppose the Markov chain is initially in the state corresponding to the third column. Using (a), (b) and appropriate theorem, find the limiting probability distribution of the Markov chain as the number of steps goes to infinity.

(e) What is the \( \lim_{n \to \infty} P^n \)?

Q4. [10 + 2 pts.] Consider the symmetric matrix

\[
A = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}
\]

which has eigenvalues \(-1, 2, 2\).
(a) Find an eigenbasis for each eigenvalue. ___/3
(b) Using (a), find an orthonormal basis of $\mathbb{R}^3$ consisting of eigenvectors of $A$ (Use Gram-Schmidt process if necessary). ___/3
(c) Write $A$ in the form $A = QDQ^{-1}$ where $D$ is a diagonal matrix and $Q$ is an orthogonal matrix. Write down only the $D$ and $Q$ matrix. ___/2
(d) Let
\[
\mathbf{x}_0 = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}
\]
and $\mathbf{x}_k = A^k \mathbf{x}_0$ for $k \geq 1$. Write $\mathbf{x}_0$ as a linear combination of the vectors in the orthonormal basis from (b) and find a formula for $\mathbf{x}_k$. The formula should involve only numbers and $k$. ___/2
(e) (Extra Credit) To which direction will $\mathbf{x}_k$ tend in the limit $k \to \infty$? ___/2

Q5. [10 pts.] Consider the vector space $P_3(\mathbb{R})$ with inner product $\langle f, g \rangle = \int_{-2}^{2} f(x)g(x)dx$.

(a) Show that $\langle \cdot, \cdot \rangle$ is a valid inner product. ___/2
(b) Prove or Disprove: $x$ is orthogonal to $x^3$. ___/1
(c) Let $W = \text{span}(\{1, x, 3x^2\})$. Find an orthonormal basis for $W$. ___/5
(d) Find the orthogonal projection of $x^3$ onto $W$. ___/2

Q6. [10 + 3 pts.] Let $\{u_1, u_2, u_3\}$ be an orthonormal subset of a complex inner product space $V$.

(a) Prove that, $\{v_1 := \frac{1}{\sqrt{2}}(u_1 + iu_3), v_2 := u_2, v_3 := \frac{1}{\sqrt{2}}(u_1 - iu_3)\}$ is also orthonormal. ___/3
(b) Find a basis for $\{u_1 + iu_3\}^\perp$ when $\dim(V) = 3$. Explain. ___/2
(c) Prove that, for any vectors $u, v \in V$ we have
\[
\|u + v\|^2 + \|u - v\|^2 = 2(\|u\|^2 + \|v\|^2).
\]
___/3
(d) Let $T : \mathbb{C}^3 \to \mathbb{C}^3$ be defined by $T(a, b, c) = (ib - 2c, ia - ic, 2a - ib)$. Determine whether $T$ is self-adjoint, normal, or neither. ___/2
(e) (Extra Credit) Let $T : \mathbb{C}^n \to \mathbb{C}^n$ be a complex linear operator satisfying
\[
T \circ T^* = \frac{1}{2}(T + T^*)
\]
Show that $T$ is diagonalizable over $\mathbb{C}$. ___/3