1. Elementary Matrices. (Sections 3.1).

- An $n \times n$ elementary matrix is obtained by performing an elementary row operation on the identity matrix $I_n$. The elementary matrix is said to be of type 1, 2, or 3 according to whether the elementary row operation performed on $I_n$ is a type 1, 2, or 3 operation, respectively.
- Let $A \in \mathcal{M}_{n \times m}(\mathbb{R})$, and suppose that $B$ is obtained from $A$ by performing an elementary row operation. Let $E$ be the elementary matrix obtained from $I_n$ by performing the same elementary row operation as that which was performed on $A$ to obtain $B$. Then $B = EA$.
- Elementary matrices are invertible and the inverse is the elementary matrix obtained by applying the inverse row operation to $I_n$.
- Any invertible matrix can be written as a product of elementary matrices.
2. Determinants. (Sections 4.1, 4.2).

- The determinant of $(A_{11}) \in M_{1 \times 1}(\mathbb{R})$ is $\det((A_{11})) = A_{11}$ and the determinant of $A \in M_{n \times n}$, $n \geq 2$, is defined recursively as

\[
\det(A) = \sum_{j=1}^{n} (-1)^{1+j} A_{1j} \det(\tilde{A}_{1j}),
\]

where $\tilde{A}_{ij}$ is the $(n-1) \times (n-1)$ matrix obtained from $A$ by deleting the $i$th row and the $j$th column.

One can also compute $\det(A)$ by taking the expansion above along any row: for any $1 \leq r \leq n$, we have

\[
\det(A) = \sum_{j=1}^{n} (-1)^{r+j} A_{rj} \det(\tilde{A}_{rj}).
\]

- (Multi-linearity of determinants) Let $n \in \mathbb{N}$, $1 \leq r \leq n$, and let $k \in \mathbb{R}$. Then

\[
\det \begin{pmatrix}
- a_1 - \\
\vdots \\
- a_{r-1} - \\
- b_r + kc_r - \\
- a_{r+1} - \\
\vdots \\
- a_n - 
\end{pmatrix} = \det \begin{pmatrix}
- a_1 - \\
\vdots \\
- a_{r-1} - \\
- b_r - \\
- a_{r+1} - \\
\vdots \\
- a_n - 
\end{pmatrix} + k \det \begin{pmatrix}
- a_1 - \\
\vdots \\
- a_{r-1} - \\
- c_r - \\
- a_{r+1} - \\
\vdots \\
- a_n - 
\end{pmatrix}.
\]

- (Determinants and row operations)

Let $A \in M_{n \times n}$ and let $R_i$ be the $i$th row of $A$. Then we have

\[
A \xrightarrow{R_i \leftrightarrow R_j} B \Rightarrow \det(B) = - \det(A),
\]

\[
A \xrightarrow{cR_i} B \Rightarrow \det(B) = c \det(A),
\]

\[
A \xrightarrow{R_i \rightarrow cR_i + R_j} B \Rightarrow \det(B) = \det(A).
\]

- (Computing $\det(A)$ using row operations) Let $A \in M_{n \times n}(\mathbb{R})$. Suppose we obtain a matrix $B \in M_{n \times n}(\mathbb{R})$ in row echelon form (hence upper triangular) after performing a sequence of row operations on $A$. Then $\det(B) = \text{the product of the diagonal entries of } B$ and we can recover $\det(A)$ using the theorem above.

- (Properties of determinants) Let $A, B \in M_{n \times n}(\mathbb{R})$. Then

1. $A$ is invertible if and only if $\det(A) \neq 0$.

2. $\det(AB) = \det(A) \det(B)$.

3. If $A$ is invertible, then $\det(A^{-1}) = \frac{1}{\det(A)}$. 
3. Diagonalization. (Sections 5.1, 5.2).

Let $V$ be an $n$-dimensional vector space over $\mathbb{R}$.

- A linear transformation $T \in \mathcal{L}(V)$ is **diagonalizable** if there exists a basis $\beta$ for $V$ for which $[T]_\beta$ is a diagonal matrix. A matrix $A \in M_{n \times n}(\mathbb{R})$ is diagonalizable if $L_A \in \mathcal{L}(\mathbb{R}^n)$ is diagonalizable.

- A vector $v \in V$ is an **eigenvector** of $T \in \mathcal{L}(V)$ if $v \neq 0$ and $T(v) = \lambda v$ for some $\lambda \in \mathbb{R}$ and $\lambda$ is the corresponding **eigenvalue**. Eigenvectors and eigenvalues of $A \in M_{n \times n}(\mathbb{R})$ are those of $L_A$.

- Let $\gamma$ be a basis for $V$. Then $T$ is diagonalizable if and only if $[T]_\gamma$ is diagonalizable.

- The **characteristic polynomial** of $A \in M_{n \times n}(\mathbb{R})$ is $f(t) = \det(A - tI_n)$.

- **(Eigenvalue test)**

  Let $A \in M_{n \times n}(\mathbb{R})$ with characteristic polynomial $f(t)$. Then $\lambda$ is an eigenvalue of $A$ if and only if $f(\lambda) = 0$; i.e., $\lambda$ is a root of $f(t)$.

- Let $\lambda$ be an eigenvalue of $A \in M_{n \times n}(\mathbb{R})$. The **eigenspace** of $\lambda$ is $E_\lambda = N(A - \lambda I_n)$ and the dimension of $E_\lambda$ is called the **geometric multiplicity** of $\lambda$. The **algebraic multiplicity** of $\lambda$ is the highest power of $t - \lambda$ that divides the characteristic polynomial of $A$.

- For any eigenvalue $\lambda$, geometric multiplicity of $\lambda$ is less than or equal to algebraic multiplicity of $\lambda$.

- **(Diagonalization criteria)** Let $A \in M_{n \times n}(\mathbb{R})$.

  - $A$ is diagonalizable if and only if there exists a basis $\beta$ for $\mathbb{R}^n$ consisting of eigenvectors of $A$.
    
    **Comment**: This criterion is normally used for theoretical purposes.

  - $A$ is diagonalizable if and only if there exist an invertible matrix $Q \in M_{n \times n}(\mathbb{R})$ and a diagonal matrix $D$ such that $D = Q^{-1}AQ$. Moreover, we can choose $Q = [I_{\mathbb{R}^n}]_\beta$, where $\beta$ is a basis for $\mathbb{R}^n$ consisting of eigenvectors of $A$ and $\gamma$ is the standard basis for $\mathbb{R}^n$, and the diagonal entries of $D$ are the eigenvalues corresponding to the eigenvectors in $\beta$.
    
    **Comment**: The expression $D = Q^{-1}AQ$ is useful for computing things like $A^k$, since $A^k = (QDQ^{-1})^k = QD^kQ^{-1}$.

  - $A$ is diagonalizable if and only if the characteristic polynomial of $A$ splits over $\mathbb{R}$ and for each eigenvalue $\lambda$ of $A$, the geometric multiplicity of $\lambda$ equals the algebraic multiplicity of $\lambda$.
    
    **Comment**: This is the diagonalization test that we use in practice.
4. Inner Product Spaces. (Section 6.1).

Let V be a vector spaces over \( \mathbb{F}(= \mathbb{R} \) or \( \mathbb{C} \)).

- An inner product on \( V \) is a map \( \langle \cdot , \cdot \rangle : V \times V \rightarrow \mathbb{F} \) such that for all \( x, y, z \in V \) and \( c \in \mathbb{F} \), the following conditions hold:
  (i) \( \langle x + z, y \rangle = \langle x, y \rangle + \langle z, y \rangle \)
  (ii) \( \langle cx, y \rangle = c \langle x, y \rangle \)
  (iii) \( \langle x, y \rangle = \langle y, x \rangle \)
  (iv) \( \langle x, x \rangle > 0 \) if \( x \neq 0 \).
- \( V \) is an inner product space if \( V \) is equipped with an (fixed) inner product.

Let \( V \) be an inner product space.

- The following conditions hold for all \( x, y, z \in V \) and \( c \in \mathbb{F} \):
  (i) \( \langle x, y + z \rangle = \langle x, y \rangle + \langle x, z \rangle \)
  (ii) \( \langle cx, y \rangle = c \langle x, y \rangle \)
  (iii) \( \langle x, 0 \rangle = \langle 0, x \rangle = 0 \)
  (iv) \( \langle x, x \rangle > 0 \) if and only if \( x \neq 0 \).
- The norm of \( v \in V \) is \( \|v\| = \sqrt{\langle v, v \rangle} \).
- For all \( x, y \in V \) and \( c \in \mathbb{F} \), we have
  (i) \( \|cx\| = |c|\|x\| \)
  (ii) \( \|x\| = 0 \) if and only if \( x = 0 \)
  (iii) \( \|x + y\| \leq \|x\| + \|y\| \) (Cauchy-Schwarz)
  (iv) \( \|x + y\| \leq \|x\| + \|y\| \) (triangle inequality).
- A subset \( S \) of \( V \) is orthogonal if \( \langle x, y \rangle = 0 \) for all distinct \( x, y \in S \) and \( S \) is orthonormal if \( S \) is orthogonal and \( \|x\| = 1 \) for all \( x \in S \).
- If \( S = \{v_1, \ldots, v_k\} \) is an orthonormal subset of \( V \), then for every \( y \in \text{span}(S) \), we have
  \[
  y = \sum_{i=1}^{k} \langle y, v_i \rangle v_i
  \]
  and the scalars \( \langle y, v_i \rangle \) are called the Fourier coefficients of \( y \) relative to \( S \). This can also be extended to the case when \( S \) is infinite.
- If \( S = \{v_1, \ldots, v_k\} \) is an orthonormal subset of \( V \), then \( S \) is linearly independent.