Math 416: Abstract Linear Algebra
Midterm 3 – Spring 2018

Date: April 18, 2018

FILL OUT THE INFORMATION IN THE BOX.

NAME: _________________________________  ID: _________________

READ THE FOLLOWING INFORMATION.

• This is a 50-minute exam.

• There is one extra credit problem. The maximum score for this exam is 50.

• Books, notes, and other aids are not allowed except for one page of cheat sheet. Collaboration is forbidden.

• Show all steps to earn full credit. Multiple answers for any problem earn ZERO credit.

• Do not unstable pages. Loose pages will be ignored.

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— Best of Luck! —
**Q1.** Let

\[ A = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}. \]

(a) Compute the characteristic polynomial of \( A \).

**Solution:** \( \det(A - tI_2) = (1 - t)(-2 - t) - 4 = t^2 + t - 6 = (t - 2)(t + 3) \).

(b) Find all the eigenvalues of \( A \).

**Solution:** 2, -3.

(c) For each eigenvalue compute the corresponding eigenspace.

**Solution:** \( E_2(A) = \mathcal{N}(A - 2I) = \{(t, t) \mid t \in \mathbb{R}\} \), \( E_{-3}(A) = \mathcal{N}(A + 3I) = \{(t, -4t) \mid t \in \mathbb{R}\} \).

(d) Find an eigenbasis of \( \mathbb{R}^2 \).

**Solution:** \( \{(1, 1), (1, -4)\} \)

(e) Now diagonalize \( A \), giving both a diagonal matrix \( D \) and an invertible matrix \( Q \) so that

\[ A = QDQ^{-1}. \]

**Solution:** \( D = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}, Q = \begin{pmatrix} 1 & 1 \\ 1 & -4 \end{pmatrix} \)

**Q2.** (a) Consider the parallelepiped \( P \) in \( \mathbb{R}^3 \) determined by the three vectors \( v = (3, 2, 1) \), \( u = (0, 2, 3) \), and \( w = (0, -2, 0) \). Compute the unsigned volume of \( P \).

**Solution:** \( \det \begin{pmatrix} 3 & 0 & 0 \\ 2 & 2 & -2 \\ 1 & 3 & 0 \end{pmatrix} = 3 \cdot 3 \cdot 2 = 18 \).

(b) Let \( A, B \in M_{3 \times 3}(\mathbb{R}) \) such that

\[ AB = \begin{pmatrix} 2 & 4 & 0 \\ 1 & 3 & -2 \\ 0 & 0 & 1 \end{pmatrix}. \]

Find \( \det(AB^2A) \). Justify each step.

**Solution:** \( \det(AB^2A) = \det(A) \det(B)^2 \det(A) = \det(AB)^2 = 4 \) since \( \det(AB) = 2 \).

**Q3.** Let \( A \in M_{n \times n}(\mathbb{R}) \) such that \( A^4 = A \).

(a) Prove that the only possible eigenvalues for the matrix \( A \) are 0 and 1.

**Solution:** Let \( \lambda \in \mathbb{R} \) be an eigenvalue of \( A \) with a corresponding eigenvector \( v \), so that \( Av = \lambda v \).

Then \( A^4 = A \) implies that \( A^4v = A^4v = \lambda^4v \) and since \( v \neq 0 \) we have \( \lambda^4 = \lambda \) or \( \lambda \in \{0, 1\} \) since \( \lambda \) is real.

(b) Prove that, \( A \) is diagonalizable iff nullity(\( A \)) + nullity(\( A - I_n \)) = \( n \).

**Solution:** Note that nullity(\( A \)) = geometric multiplicity of 0 and nullity(\( A - I_n \)) = geometric multiplicity of 1.

**Only if part:** Since \( A \) is diagonalizable and the only eigenvalues of \( A \) are 0, 1 we have algebraic multiplicity of 0 = geometric multiplicity of 0 = algebraic multiplicity of 1 = geometric multiplicity of 1, which proves the result.

**If part:** We break into 3 cases, i) nullity(\( A \)) = 0, ii) nullity(\( A - I \)) = 0 and iii) nullity(\( A \)) > 0, nullity(\( A - I \)) > 0. In all cases we have sum of all algebraic multiplicities = \( n \); geometric
multiplicity of $0 = \text{algebraic multiplicity of } 0$, and geometric multiplicity of $1 = \text{algebraic multiplicity of } 1$. Thus the characteristic polynomial of $A$ completely factorizes over $\mathbb{R}$ and algebraic and geometric multiplicities are the same for all eigenvalues.

**Q4.** (a) The island of Madagascar is home to almost all of the world’s lemurs, a family of primitive non-ape primates. Divide Madagascar into 2 regions, the North and the South, along the Mania river. Suppose that in any given year, $1/6$ of the lemurs living in the North move to the South, and $1/3$ of those in the South move to the North. Give the transition matrix $P$ for the corresponding Markov chain.

$$P = \begin{pmatrix} 1/6 & 1/3 \\ 5/6 & 2/3 \end{pmatrix}.$$  

**Solution:**

$$P = \begin{pmatrix} 5/6 & 1/3 \\ 1/6 & 2/3 \end{pmatrix}.$$  

(b) Find a probability vector $\mathbf{u}$ in $E_1(P)$.

**Solution:** We find $E_1(P) = \{ (2t, t) \mid t \in \mathbb{R} \}$. Thus $\mathbf{u} = (2/3, 1/3)$.

(c) What is

$$\lim_{m \to \infty} P^m = \begin{pmatrix} \end{pmatrix}.$$  

**Solution:**

$$\lim_{m \to \infty} P^m = \begin{pmatrix} 2/3 & 2/3 \\ 1/3 & 1/3 \end{pmatrix}.$$  

(d) Let $V$ be an inner product space with norm $\| \cdot \|$ and $\mathbf{v}, \mathbf{u} \in V$ be two vectors such that

$$\| \mathbf{u} \| = 2, \| \mathbf{v} \| = 2, \| \mathbf{u} - \mathbf{v} \| = 3.$$  

Find

$$\| \mathbf{u} + \mathbf{v} \| = \_ \_ \_ \_ \_ \_ \_ \_ \_ \_ .$$  

**Solution:** Check that $\| \mathbf{u} + \mathbf{v} \|^2 + \| \mathbf{u} - \mathbf{v} \|^2 = 2 \| \mathbf{u} \|^2 + 2 \| \mathbf{v} \|^2$. Thus $\| \mathbf{u} + \mathbf{v} \|^2 = 2(4^2 + 4^2) - 3^2 = 7$ and $\| \mathbf{u} + \mathbf{v} \| = \sqrt{7}$.

**Q5.** Circle true or false as appropriate; you DO NOT need to provide any justification.

(a) A square matrix is diagonalizable if and only if its characteristic polynomial splits completely and it has distinct eigenvalues. 

[ F ]

(b) If $E, F$ are elementary matrices of size $n \times n$, then so is $EF$. 

[ F ]
(c) If a $4 \times 4$ matrix has 4 distinct eigenvalues, it is diagonalizable.  


\[ \text{T} \]  

(d) For $A, B \in \mathcal{M}_{n\times n}(\mathbb{R})$, we have $\det(A + B) = \det(A) + \det(B)$.  


\[ \text{F} \]  

(e) There exists a transition matrix where $-3$ is an eigenvalue.  


\[ \text{F} \]  

(f) A transition matrix always has at least one eigenvalue.  


\[ \text{T} \]  

(g) The formula $\langle x, y \rangle = 2x_1 y_1$ defines an inner product on $\mathbb{R}^2$.  


\[ \text{F} \]  

(h) The formula $\langle f, g \rangle = \int_0^1 f(t) g(t) dt$ defines an inner product on $P_2(\mathbb{R})$.  


\[ \text{T} \]  

Extra Credit Find a $2 \times 2$ matrix $A$ such that $A^3 = A$ but $A$ has no eigenvalues. Explain.

Solution: We claim that there is no such matrix.

Note that $A$ must be invertible, otherwise $0$ is an eigenvalue. Thus $A^2 = I$. The matrix

\[ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \]

satisfies $A^2 = I$ implies either $b = c = 0, a^2 = d^2 = 1$ or $a = -d, bc = 1 - a^2$. In the first case the matrix is diagonal hence is diagonalizable, in the second case \( \begin{pmatrix} a & b \\ (1-a^2)/b & -a \end{pmatrix} \) has eigenvalues $\pm 1$ as the characteristic polynomial is $t^2 - 1$.

or

$A^3 = A$ implies $A(A - I)(A + I) = A(A^2 - I) = 0$. Thus at least one of the matrices $A, A - I, A + I$ is non-invertible. Which means $A$ must have at least one real eigenvalue in $\{0, 1, -1\}$. 

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