Math 416: Abstract Linear Algebra
Midterm 3 – Spring 2018
Date: April 18, 2018

FILL OUT THE INFORMATION IN THE BOX.

NAME: ________________________________     ID: ____________________

READ THE FOLLOWING INFORMATION.

• This is a 50-minute exam.

• There is one extra credit problem. The maximum score for this exam is 50.

• Books, notes, and other aids are not allowed except for one page of cheat sheet. Collaboration is forbidden.

• Show all steps to earn full credit. Multiple answers for any problem earn ZERO credit.

• Do not unstable pages. Loose pages will be ignored.

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— Best of Luck! —
(This page intentionally left blank. You can use it for scratch work.)
Q1. Let

\[ A = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}. \]

(a) Compute the characteristic polynomial of \( A \).

(b) Find all the eigenvalues of \( A \).

(c) For each eigenvalue compute the corresponding eigenspace.

(d) Find an eigenbasis of \( \mathbb{R}^2 \).

(e) Now diagonalize \( A \), giving both a diagonal matrix \( D \) and an invertible matrix \( Q \) so that

\[ A = QDQ^{-1}. \]
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Q2. (a) Consider the parallelepiped $P$ in $\mathbb{R}^3$ determined by the three vectors $v = (3, 2, 1)$, $u = (0, 2, 3)$, and $w = (0, -2, 0)$. Compute the unsigned volume of $P$.

(b) Let $A, B \in \mathcal{M}_{3 \times 3}(\mathbb{R})$ such that

$$AB = \begin{pmatrix} 2 & 4 & 0 \\ 1 & 3 & -2 \\ 0 & 0 & 1 \end{pmatrix}.$$ 

Find $\det(AB^2A)$. Justify each step.
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Q3. Let $A \in \mathcal{M}_{n \times n}(\mathbb{R})$ such that $A^4 = A$.

(a) Prove that the only possible eigenvalues for the matrix $A$ are 0 and 1. 

(b) Prove that, $A$ is diagonalizable iff $\text{nullity}(A) + \text{nullity}(A - I_n) = n$. 

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Q4. (a) The island of Madagascar is home to almost all of the world’s lemurs, a family of primitive non-ape primates. Divide Madagascar into 2 regions, the North and the South, along the Mania river. Suppose that in any given year, 1/6 of the lemurs living in the North move to the South, and 1/3 of those in the South move to the North. Give the transition matrix $P$ for the corresponding Markov chain.

$$P = \begin{pmatrix} \ & \ \\ \end{pmatrix}.$$  

(b) Find a probability vector $u$ in $E_1(P)$.  

(c) What is

$$\lim_{m \to \infty} P^m = \begin{pmatrix} \ & \ \\ \end{pmatrix}.$$  

(d) Let $V$ be an inner product space with norm $\|\cdot\|$ and $v, u \in V$ be two vectors such that

$\|u\| = 2, \|v\| = 2, \|u - v\| = 3$.

Find

$\|u + v\| =$
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Q5. Circle true or false as appropriate; you **DO NOT** need to provide any justification.

(a) A square matrix is diagonalizable if and only if its characteristic polynomial splits completely and it has distinct eigenvalues.

(b) If \( E, F \) are elementary matrices of size \( n \times n \), then so is \( EF \).

(c) If a \( 4 \times 4 \) matrix has 4 distinct eigenvalues, it is diagonalizable.

(d) For \( A, B \in M_{n \times n}(\mathbb{R}) \), we have \( \det(A + B) = \det(A) + \det(B) \).

(e) There exists a transition matrix where \(-3\) is an eigenvalue.

(f) A transition matrix always has at least one eigenvalue.

(g) The formula \( \langle x, y \rangle = 2x_1 y_1 \) defines an inner product on \( \mathbb{R}^2 \).

(h) The formula \( \langle f, g \rangle = \int_0^1 f(t)g(t)\,dt \) defines an inner product on \( P_2(\mathbb{R}) \).
(This page intentionally left blank. You can use it for scratch work.)
Extra Credit  Find a $2 \times 2$ matrix $A$ such that $A^3 = A$ but $A$ has no eigenvalues. Explain.
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