Math 416: Abstract Linear Algebra
Midterm 2 – Spring 2018
Date: March 14, 2018

FILL OUT THE INFORMATION IN THE BOX.

NAME: ________________________________  netID: ________________

READ THE FOLLOWING INFORMATION.

• This is a 50-minute exam.

• There is one extra credit problem. The maximum score for this exam is 50.

• Books, notes, and other aids are not allowed except for one page of cheat sheet. Collaboration is forbidden.

• Show all steps to earn full credit. Multiple answers for any problem earn ZERO credit.

• Do not unstaple pages. Loose pages will be ignored.

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— Best of Luck! —
Q1. Consider the left-multiplication linear transformation \( L_A : \mathbb{R}^3 \to \mathbb{R}^2 \) where \( A = \begin{pmatrix} 2 & 3 & 1 \\ 6 & 9 & 3 \end{pmatrix} \)

(a) Compute \( L_A(1, -1, 2) \). 

____/1

(b) Find a basis for \( \text{N}(L_A) \). 

____/3

(c) Find a basis for \( \text{R}(L_A) \). 

____/3

(d) Directly verify that the dimension theorem holds for \( L_A \). 

____/2

(e) Let \( \alpha, \beta \) be the standard bases for \( \mathbb{R}^3 \) and \( \mathbb{R}^2 \), respectively. What is \([L_A]_\alpha^\beta\)? 

____/1
Q2. Define the linear transformation $T : \mathcal{P}_3(\mathbb{R}) \to \mathcal{P}_2(\mathbb{R})$ by $T(p(x)) = p''(x)$.

(a) Let $\beta, \gamma$ be the standard bases for $\mathcal{P}_3(\mathbb{R}), \mathcal{P}_2(\mathbb{R})$, respectively. Find $[T]_{\beta}^\gamma$.

(b) Find nullity$(T)$.

(c) Find rank$(T)$.

(d) Let $\alpha = \{1, x - 1, x^2 - x, x^3 - x^2\}$ be another basis for $\mathcal{P}_3(\mathbb{R})$. Find the change of basis matrix $[I_{\mathcal{P}_3(\mathbb{R})}]_{\alpha}^\beta$.

(e) What is the relation among $[T]_{\beta}^\gamma$, $[T]_{\alpha}^\gamma$ and $[I_{\mathcal{P}_3(\mathbb{R})}]_{\alpha}^\beta$. 

___/3

___/2

___/2

___/2

___/1
(This page intentionally left blank. You can use it for scratch work.)
Q3. Let $T : V \rightarrow W$ be a linear transformation and $\beta = \{v_1, v_2, \ldots, v_n\}$ be a basis for $V$. Let $\gamma := \{w_1, w_2, \ldots, w_n\}$ where $w_i = T(v_i)$ for $i = 1, 2, \ldots, n$.

(a) Prove that, $\gamma$ is a basis for $W$ if $T$ is invertible.

(b) Prove that, $T$ is invertible if $\gamma$ is a basis for $W$. 
(This page intentionally left blank. You can use it for scratch work.)
Q4. (a) Let $A \in \mathcal{M}_{n \times n}(\mathbb{R})$. Prove that, $\text{nullity}(A) = \text{nullity}(A^t)$.

(b) Let $A, B \in \mathcal{M}_{n \times n}(\mathbb{R})$ be invertible matrices. Prove that, $AB$ is invertible and find $(AB)^{-1}$ in terms of $A^{-1}$ and $B^{-1}$. 
Q5. Circle true or false as appropriate; you **DO NOT** need to provide any justification.

(a) The vector spaces $\mathcal{M}_{2 \times 3}(\mathbb{R})$ and $\mathcal{P}_5(\mathbb{R})$ are isomorphic.

   [ ] True  [ ] False

(b) If $V$ and $W$ are finite dimensional vector spaces, then so is $\mathcal{L}(V, W)$.

   [ ] True  [ ] False

(c) A transformation $T$ from $\mathbb{R}^2$ to $\mathbb{R}^2$ maps $(1, 0)$ to $(2, 0)$ and $(3, 0)$ to $(6, 0)$. $T$ must be a linear transformation.

   [ ] True  [ ] False

(d) If a linear transformation $T : V \to V$ from a finite dimensional vector space to itself is onto, then $T$ is also one-to-one.

   [ ] True  [ ] False

(e) If the RREF of $A$ is $R$, the RREF of $\begin{pmatrix} A & A \\ A & A \end{pmatrix}$ is $\begin{pmatrix} R & R \\ 0 & 0 \end{pmatrix}$.

   [ ] True  [ ] False

(f) For all $A \in \mathcal{M}_{n \times n}(\mathbb{R})$, one has $\mathcal{R}(L_A) = \text{RowSpace}(A)$.

   [ ] True  [ ] False

(g) For all square matrices $A$, one has $\det(-A) = -\det(A)$.

   [ ] True  [ ] False

(h) Let $A$ be a square matrix with $\mathcal{N}(A) = \{0\}$. Then $\mathcal{N}(A^2) = \{0\}$.

   [ ] True  [ ] False
(This page intentionally left blank. You can use it for scratch work.)
Q6. (a) Find $\det \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 2 & 0 & 1 \end{pmatrix}$. State which method you used. ___/4

(b) Use your above result to find the determinant of the following matrix. State your method. ___/2

$$\det \begin{pmatrix} 1 & 2 & 2 & 1 \\ 2 & 1 & -3 & 2 \\ 0 & 0 & 1 & 0 \\ 2 & 0 & -1 & 1 \end{pmatrix} = \underline{\hphantom{123456789012345678901234567890}}.$$ 

Extra Credit. Let $AB \in \mathcal{M}_{n \times n}(\mathbb{R})$ be such that $AB = I_n$. Prove that $BA = I_n$. ___/4
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