Homework 9

MATH 416: Abstract Linear Algebra

Due date: April 27, 2018

Textbooks: In the assignment, the two texts are abbreviated as follows:


1. Section 6.2 of [FIS], Problem 2 parts (b, c, g).

Solution:

(b) Pick \( w_1 = (0,0,1), w_2 = (0,1,1), \) and \( w_3 = (1,1,1) \) and get the answer \( \beta = \{(0,0,1), (0,1,0), (1,0,0)\} \).

And we also know that the Fourier coefficients of \( x \) relative to \( \beta \) are \( 1, 0, 1 \).

(c) The basis is \( \beta = \{1, \sqrt{3}(2x - 1), \sqrt{5}(6x^2 - 6x + 1)\} \) and the Fourier coefficients are \( \frac{3}{2}, \frac{\sqrt{3}}{6}, 0 \).

(g) The basis is \( \beta = \{\left(\frac{1}{2}, \frac{5}{6}\right), \left(-\frac{\sqrt{2}}{3}, \frac{\sqrt{2}}{3}\right), \left(\frac{1}{\sqrt{2}}, -\frac{1}{3}\right)\} \)

and the Fourier coefficients are \( 24, 6\sqrt{2}, -9\sqrt{2} \).

2. Section 6.2 of [FIS], Problems 6 and 13 (a-c).

Solution:

6. Apply Theorem 6.6 to \( x \) in \( V \). We know that \( x \) could be uniquely written as \( u + v \) with \( u \in W \) and \( v \in W^\perp \). Since \( x \notin W \), we have \( v \neq 0 \). Pick \( y = v \). We have \( \langle x, y \rangle = \langle u, v \rangle + \langle v, v \rangle = \|v\|^2 > 0 \).

13.(a) If \( x \in S^\perp \) then we have that \( x \) is orthogonal to all elements of \( S \), so are all elements of \( S_0 \). Hence we have \( x \in S_0^\perp \).

13.(b) If \( x \in S \), we have that \( x \) is orthogonal to all elements of \( S^\perp \). This means \( x \) is also an element in \( (S^\perp)^\perp \) and thus \( S \subseteq (S^\perp)^\perp \). This implies that

\[
\text{span}(S) \subseteq (S^\perp)^\perp
\]

since \( \text{span}(S) \) is the smallest subspace containing \( S \) and every orthogonal complement is a subspace.

13.(c) By the previous argument, we already have that \( W \subseteq (W^\perp)^\perp \). For the converse, if \( x \notin W \), we may find, by Problem 6), \( y \in W^\perp \) and \( \langle x, y \rangle \neq 0 \) and hence \( x \notin (W^\perp)^\perp \). This means that \( W \supseteq (W^\perp)^\perp \).

3. Section 6.2 of [FIS], Problem 7.

Solution: The necessity comes from the definition of orthogonal complement, since every element in \( \beta \) is an element in \( W \). For the sufficiency, assume that \( \langle z, v \rangle = 0 \) for all \( v \in \beta \). Since \( \beta \) is a basis, every element in \( W \) could be written as \( \sum_{i=1}^{k} a_i v_i \) where \( a_i \) is some scalar and \( v_i \) is element in \( \beta \). So we have

\[
\langle z, \sum_{i=1}^{k} a_i v_i \rangle = \sum_{i=1}^{k} a_i \langle z, v_i \rangle = 0.
\]

Hence \( z \) is an element in \( W^\perp \).
4. Section 6.2 of [FIS], Problem 8.

**Solution:** We apply induction on $n$. When $n = 1$, the Gram-Schmidt process always preserve the first vector. Suppose the statement holds for $n \leq k$. Consider the a orthogonal set of nonzero vectors $\{w_1, w_2, \ldots, w_k\}$. By induction hypothesis, we know that the vectors $v_i = w_i$ for $i = 1, 2, \ldots, k-1$, where $v_i$ is the vector derived from the process. Now we apply the process the find

$$v_n = w_n - \sum_{i=1}^{k-1} \frac{\langle w_n, v_i \rangle}{\|v_i\|^2} v_i = w_n - 0 = w_n.$$

So we get the desired result.

5. Section 6.2 of [FIS], Problem 11.

**Solution:** Since we did not discuss complex matrices before this week, we will only work with real matrices so that $A^* = A^t$, but the same proof works for complex matrices. Use the fact $(AA^t)_{ij} = \langle v_i, v_j \rangle$ for all $i$ and $j$, where $v_i$ is the $i$-th row vector of $A$.

6. Section 6.3 of [FIS], Problem 12.

**Solution:**

(a) If $x \in R(T^* \perp)$ we have $0 = \langle x, T^*(y) \rangle = \langle T(x), y \rangle$ for all $y$. Thus we have $T(x) = 0$ or $x \in \mathcal{N}(T)$. Conversely, if $x \in \mathcal{N}(T)$, we have $\langle x, T^*(y) \rangle = \langle T(x), y \rangle = 0$ for all $y$. This means that $x$ is an element in $R(T^* \perp)$.

(b) By Exercise 6.2.13(c) we have $\mathcal{N}(T)^\perp = (R(T^* \perp))^\perp = R(T^*)$.

7. Section 6.3 of [FIS], Problem 14.

**Solution:** It is linear since

$$T(cx_1 + x_2) = \langle cx_1 + x_2, y \rangle z = c\langle x_1, y \rangle z + \langle x_2, y \rangle z = cT(x_1) + T(x_2).$$

On the other hand, we have

$$\langle T(u), v \rangle = \langle (u, y)z, v \rangle = \langle u, y \rangle \langle z, v \rangle = \langle u, (v, z)y \rangle$$

for all $u$ and $v$. So we have

$$T^*(x) = \langle x, z \rangle y.$$