Homework 6

MATH 416: Abstract Linear Algebra

Due date: March 28, 2018

Textbooks: In the assignment, the two texts are abbreviated as follows:


1. Section 4.2 of [FIS], Problem 1.

Solution:
(a) False. For \( A \in \mathcal{M}_{n \times n}(\mathbb{R}), c \in \mathbb{R} \) we have \( \det(cA) = c^n \det(A) \).
(b) True. This is Theorem 4.4.
(c) True. This is the Corollary after Theorem 4.4.
(d) True. This is Theorem 4.5.
(e) False. For example, the determinant of \(
\begin{pmatrix}
2 & 0 \\
0 & 1 \\
\end{pmatrix}
\) is 2 but \( \det(I_2) = 1 \).
(f) False. We have that \( \det \left( \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \right) = 1 \neq 2 = 2 \det \left( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \).
(g) False. For example, the determinant of identity matrix is 1.
(h) True. See Exercise 4.2.23.

2. Section 4.2 of [FIS], Problem 30.

Solution: We can interchange the \( i \)-th row and the \((n + 1 - i)\)-th row for all \( i = 1, 2, \ldots, \lfloor n/2 \rfloor \) where \( \lfloor x \rfloor \) is the integer part of \( x \). Each process contribute \(-1\) one time. So we have that \( \det(B) = (-1)^{\lfloor n/2 \rfloor} \det(A) \).

3. Prove the following result that was used in class. Suppose \( E \) is the elementary matrix obtained from \( I_n \) by the row operation \( R \), that is, \( I_n \overset{R}{\rightarrow} E \). Prove that for all \( A \in \mathcal{M}_{n \times n}(\mathbb{R}) \) one has \( A \overset{R}{\rightarrow} EA \). Said another way, left-multiplication by \( E \) implements the row operation that built \( E \) in the first place.

Solution: Let \( A \in \mathcal{M}_{n \times n}(\mathbb{R}) \) be a square matrix with rows given by \( u_1, u_2, \ldots, u_n \), i.e., \( A = \begin{pmatrix}
u_1 \\ u_2 \\ \vdots \\ u_n \\
\end{pmatrix} \). Let \( e_1, e_2, \ldots, e_n \) be the rows of the identity matrix \( I_n \). It’s enough to check the following matrix multiplication is right for \( 1 \leq i < j \leq n, c \in \mathbb{R} \).

\[
\begin{pmatrix}
e_1 \\ e_2 \\ \vdots \\ e_j \\ e_i \\ \vdots \\ e_n \\
\end{pmatrix}
\begin{pmatrix}
u_1 \\ u_2 \\ \vdots \\ u_i \\ \vdots \\ u_j \\ \vdots \\ u_n \\
\end{pmatrix}
= 
\begin{pmatrix}
u_1 \\ u_2 \\ \vdots \\ u_i \\ \vdots \\ u_j \\ \vdots \\ u_n \\
\end{pmatrix}
\]
(b) \[
\begin{pmatrix}
e_1 & u_1 \\
e_2 & u_2 \\
\vdots & \vdots \\
e_i & u_i \\
\vdots & \vdots \\
e_n & u_n \\
\end{pmatrix}
= 
\begin{pmatrix}
u_1 \\
u_2 \\
\vdots \\
u_i \\
\vdots \\
u_n \\
\end{pmatrix}
\]

(c) \[
\begin{pmatrix}
e_1 & u_1 \\
e_2 & u_2 \\
\vdots & \vdots \\
e_i & u_i \\
\vdots & \vdots \\
e_n & u_n \\
\end{pmatrix}
= 
\begin{pmatrix}
u_1 \\
u_2 \\
\vdots \\
u_i + cu_j \\
\vdots \\
u_n \\
\end{pmatrix}
\]

One can check that by comparing values in \((k,l)\)-th position for \(1 \leq k, l \leq n\).

4. Prove that if \(A, B \in \mathcal{M}_{n \times n}(\mathbb{R})\) are similar matrices then \(\det(A) = \det(B)\).

**Solution:** If \(A, B \in \mathcal{M}_{n \times n}(\mathbb{R})\) are similar matrices then there exists an invertible matrix \(Q\) such that 

\[A = Q^{-1}BQ.\]

Thus \(\det(A) = \det(Q^{-1}BQ) = \det(Q^{-1})\det(B)\det(Q) = \det(Q)\det(Q^{-1})\det(B) = \det(QQ^{-1})\det(B) = 1 \cdot \det(B) = \det(B)\).

5. A matrix \(Q \in \mathcal{M}_{n \times n}(\mathbb{R})\) is called orthogonal if \(QQ^t = I_n\).

(a) Prove that if \(Q\) is orthogonal then \(\det(Q) = \pm 1\).

(b) Give examples of orthogonal matrices for \(n = 2\) with both possible values of the determinant.

**Solution:**

(a) If \(Q\) is orthogonal, then \(QQ^t = I_n\) and \(\det(QQ^t) = \det(I_n) = 1\). Now, \(\det(Q^t) = \det(Q)\) and \(\det(QQ^t) = \det(Q)\det(Q^t)\). Thus we have \(\det(Q)^2 = 1\) or \(\det(Q) = \pm 1\).

(b) For \(Q = I_2\), we have \(QQ^t = I_2\) and \(\det(Q) = 1\). For \(Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\) we have \(\det(Q) = -1\) and \(QQ^t = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = I_2\).

6. Suppose \(A, B \in \mathcal{M}_{n \times n}(\mathbb{R})\) satisfy \(AB = I_n\).

(a) Use the determinant to prove that \(A\) is invertible.

(b) Prove or disprove: \(B = A^{-1}\).

**Solution:**

(a) We have \(\det(AB) = \det(A)\det(B)\). Thus \(AB = I_n\) implies that \(\det(A)\det(B) = 1\). Hence \(\det(A) \neq 0\) and \(\det(B) \neq 0\) and both \(A, B\) are invertible.

(b) We have \(B = A^{-1}\). Since \(A\) is invertible, \(AB = I_n\) implies that \(A^{-1}(AB) = A^{-1}I_n\) or \(B = A^{-1}\).

7. Let \(M\) be a square matrix of the form \(\begin{pmatrix} A & B \\ 0 & I_n \end{pmatrix}\), for some matrix \(A, B\) and the identity matrix \(I_n\). Using induction on \(n\), prove that \(\det(M) = \det(A)\).

**Solution:** The matrix \(A\) must be a square matrix, for \(M\) to be a square matrix. Let \(k\) be the number of rows of \(A\).
The statement is true for \( n = 1 \), by expanding along the last row with only one entry being 1. Suppose it is true for \( n - 1 \). For \( n \), expanding along the last row with only one entry being 1, we have

\[
\det \begin{pmatrix} A & B \\ 0 & I_n \end{pmatrix} = (-1)^{k+n+k+n} \cdot 1 \cdot \det \begin{pmatrix} A & B' \\ 0 & I_{n-1} \end{pmatrix} = \det(A)
\]

where \( B' \) is the matrix obtained from \( B \) by deleting the last column and the second equality follows by induction hypothesis.