Homework 5a

MATH 416: Abstract Linear Algebra

Due date: Will not be graded

Textbooks: In the assignment, the two texts are abbreviated as follows:


1. Suppose \( T : V \to W \) is a linear transformation between finite-dimensional vector spaces, and let \( \beta = \{v_1, \ldots, v_n\} \) be a basis for \( V \). Prove that \( T \) is an isomorphism if and only if \( \gamma = \{w_1, \ldots, w_n\} \) where \( w_i = T(v_i) \) is a basis for \( W \).

Hint: We did part of this in class.

Solution:

If: If \( \gamma \) is a basis for \( W \), then \( T \) is onto as \( \text{span}(T(v_1), T(v_2), \ldots, T(v_n)) = \text{span}(\gamma) = W \). Moreover, we claim that \( T(u) = 0 \) implies \( u = 0 \). We can write \( u = a_1v_1 + a_2v_2 + \cdots + a_nv_n \) as \( \beta \) is a basis for \( V \). Then \( a_1w_1 + a_2w_2 + \cdots + a_nw_n = 0 \) which implies that \( a_1 = a_2 = \cdots = a_n = 0 \) or \( v = 0 \). Thus \( T \) is 1-1 and is an isomorphism.

Only if: If \( T \) is an isomorphism, then \( T \) is one-to-one and onto. Thus \( W = \text{span}(T(v_1), T(v_2), \ldots, T(v_n)) = \text{span}(\gamma) \). Moreover, \( \gamma \) is linearly independent, as \( a_1w_1 + a_2w_2 + \cdots + a_nv_n = 0 \) implies that \( T(a_1v_1 + a_2v_2 + \cdots + a_nv_n) = a_1T(v_1) + a_2T(v_2) + \cdots + a_nT(v_n) = 0 \) and thus \( a_1v_1 + a_2v_2 + \cdots + a_nv_n = 0 \) since \( T \) is 1-1. This implies that \( a_1 = a_2 = \cdots = a_n = 0 \). Thus \( \gamma \) is a basis of \( W \).

2. Section 2.5 of [FIS], Problem 1.

Solution:

(a) False. It should be \([x'_j]_{\beta'}\).
(b) True. This is Theorem 2.22.
(c) True. This is Theorem 2.23.
(d) False. It should be \(B = Q^{-1}AQ\).
(e) True. This is the instant result of the definition of similar and Theorem 2.23.

3. Section 2.5 of [FIS], Problem 2 and Problem 3 (c) and (d).

Solution: For problem 2. and 3., just calculate \([I]^{\beta}_{\beta'}\).

2.(a) \( \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \).
2.(b) \( \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix} \).
2.(c) \( \begin{pmatrix} 3 & -1 \\ 5 & -2 \end{pmatrix} \).
2.(d) \( \begin{pmatrix} 2 & -1 \\ 5 & -4 \end{pmatrix} \).
3.(c) \( \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ -3 & 2 & 1 \end{pmatrix} \).
3. (d) \[
\begin{pmatrix}
2 & 1 & 1 \\
3 & -2 & 1 \\
-1 & 3 & 1
\end{pmatrix}.
\]

4. Section 2.5 of [FIS], Problem 6 (a) and (c).
   
   **Solution:** Let $\alpha$ be the standard basis (of $\mathbb{R}^2$ or $\mathbb{R}^3$). We have that $A = [L_A]_\alpha$ and hence $[L_A]_\beta = [I]_\beta^\alpha [L_A]_\alpha [I]_\alpha^\beta$. So now we can calculate $[L_A]_\beta$ and $Q = [I]_\beta^\alpha$ and $Q^{-1} = [I]_\alpha^\beta$.

6. (a) $[L_A]_\beta = \begin{pmatrix} 6 & 11 \\ -2 & -4 \end{pmatrix}$ and $Q = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$.

6. (c) $[L_A]_\beta = \begin{pmatrix} 2 & 2 & 2 \\ -2 & -3 & -4 \\ 1 & 1 & 2 \end{pmatrix}$ and $Q = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \end{pmatrix}$.

5. Section 2.5 of [FIS], Problem 7.
   
   **Solution:** We may let $\beta$ be the standard basis and $\alpha = \{(1, m), (-m, 1)\}$ be another basis for $\mathbb{R}^2$.

   (a) We have $[T]_\alpha = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $Q^{-1} = [I]_\alpha^\beta = \begin{pmatrix} 1 & -m \\ m & 1 \end{pmatrix}$. We can also calculate that $Q = [I]_\beta^\alpha = \frac{1}{1+m^2} \begin{pmatrix} 1 \\ -m \end{pmatrix}$. So finally we get
   
   $[T]_\beta = Q^{-1}[T]_\alpha Q = \frac{1}{1+m^2} \begin{pmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{pmatrix}$,
   
   i.e.,
   
   $T(x, y) = \left( \frac{x+2ym-xm^2}{1+m^2}, \frac{-y+2xm+ym^2}{1+m^2} \right)$.

   (b) Similarly, we have $[T]_\alpha = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and with the same $Q, Q^{-1}$ matrix. Thus we get
   
   $[T]_\beta = Q^{-1}[T]_\alpha Q = \frac{1}{1+m^2} \begin{pmatrix} 1 \\ m \end{pmatrix}$,
   
   i.e.,
   
   $T(x, y) = \left( \frac{x+ym}{1+m^2}, \frac{xm+ym^2}{1+m^2} \right)$.

6. Compute the determinants of the following matrices:
   
   (a) $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, (b) $\begin{pmatrix} -3 & 2 \\ 2 & -3 \end{pmatrix}$, (c) $\begin{pmatrix} 4 & 5 \\ -4 & -5 \end{pmatrix}$.
   
   **Solution:**
   
   (a) $1 \cdot 4 - 2 \cdot 3 = -2$.
   
   (b) 5.
   
   (c) 0.

7. Suppose $A \in \mathcal{M}_{2 \times 2}(\mathbb{R})$.
   
   (a) Show that $\det(A) = \det(A^t)$.
   
   (b) Show that if $B$ is obtained from $A$ by swapping the two rows, then $\det(B) = -\det(A)$.
   
   (c) How does the determinant change if instead you swap the columns of $A$?
   
   (d) If $B$ is also in $\mathcal{M}_{2 \times 2}(\mathbb{R})$, prove that $\det(AB) = \det(A) \det(B)$.
   
   **Solution:** Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $\det(A) = ad - bc$. 

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(a) $A^t = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ with $\det(A^t) = ad - bc$.

(b) $B = \begin{pmatrix} c & d \\ a & b \end{pmatrix}$ with $\det(B) = bc - ad = -\det(A)$.

(c) We get $-\det(A)$.

(d) Let $B = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$ with $\det(B) = ps - qr$. We check that

$$AB = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} ap + br & aq + bs \\ cp + dr & cq + ds \end{pmatrix}$$

with $\det(AB) = (ap + br)(cq + ds) - (aq + bs)(cp + dr) = (ad - bc)(ps - qr) = \det(A) \det(B)$.

8. Section 4.1 of [FIS], Problem 10.

**Solution:**

(a) We can directly check that

$$CA = \begin{pmatrix} A_{11}A_{22} - A_{12}A_{21} & 0 \\ 0 & A_{11}A_{22} - A_{12}A_{21} \end{pmatrix} = (A_{11}A_{22} - A_{12}A_{21}) \cdot I = [\det(A)]I$$

and

$$AC = \begin{pmatrix} A_{11}A_{22} - A_{12}A_{21} & 0 \\ 0 & A_{11}A_{22} - A_{12}A_{21} \end{pmatrix} = CA.$$

(b) We calculate that $\det(C) = A_{22}A_{11} - (-A_{12})(-A_{21}) = A_{11}A_{22} - A_{12}A_{21} = \det(A)$.

(c) Since the transpose matrix of $A$ is $A^t = \begin{pmatrix} A_{11} & A_{12} \\ A_{12} & A_{22} \end{pmatrix}$, the corresponding classical adjoint would be

$$\begin{pmatrix} A_{22} & -A_{21} \\ -A_{12} & A_{11} \end{pmatrix} = C^*.$$

(d) If $A$ is invertible, we have $\det(A) \neq 0$. So we can write from (a),

$$[\det(A)]^{-1}CA = [\det(A)]^{-1}AC = I,$$

and get the desired result.

9. Section 4.2 of [FIS], Problems 5 and 11.

**Solution:**

4.2.5. The determinant should be $-12$ by following processes:

$$\begin{vmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ 2 & 3 & 0 \end{vmatrix} = 0 \cdot \begin{vmatrix} 0 & -3 \\ 2 & 0 \end{vmatrix} - 1 \cdot \begin{vmatrix} -1 & -3 \\ 2 & 0 \end{vmatrix} + 2 \cdot \begin{vmatrix} -1 & 0 \\ 2 & 3 \end{vmatrix} = 0 - 1(6) + 2(-3) = -12.$$  

4.2.11. The determinant should be $-3$.

10. Section 4.2 of [FIS], Problem 21.

**Solution:** The determinant should be $95$. 