Homework 5

MATH 416: Abstract Linear Algebra

Due date: March 7, 2018

Textbooks: In the assignment, the two texts are abbreviated as follows:


1. Section 2.3 of [FIS], Problem 1.

Solution:

(a) False. It should be \([UT]_{\alpha}^{\gamma} = [U]_{\alpha}^{\gamma}[T]_{\alpha}^{\beta}\).

(b) True. That’s Theorem 2.14.

(c) False. In general \(\beta\) is not a basis for \(V\).

(d) True. That’s Theorem 2.12.

(e) False. It will be true when \(\alpha = \beta\).

(f) False. We have \((\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix})^2 = I\).

(g) False. \(T\) is a transformation from \(V\) to \(W\) but \(L_A\) can only be a transformation from \(\mathbb{R}^m\) to \(\mathbb{R}^n\).

(h) False. We have \((\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix})^2 = 0\).

(i) True. That’s Theorem 2.15.

(j) True. Since \(\delta_{ij} = 1\) only when \(i = j\), we have \(A_{ij} = \delta_{ij}\).

2. Section 2.3 of [FIS], Problem 2.

Solution:

(a) We have \(A(2B + 3C) = \begin{pmatrix} 20 & -9 & 18 \\ 5 & -10 & 8 \end{pmatrix}\) and \((AB)D = A(BD) = \begin{pmatrix} 29 \\ -26 \end{pmatrix}\).

(b) \(A^t = \begin{pmatrix} 2 & -3 & 4 \\ 5 & 1 & 2 \end{pmatrix}\), \(A^tB = \begin{pmatrix} 23 & 19 & 0 \\ 16 & -1 & 10 \end{pmatrix}\), \(BC^t = \begin{pmatrix} 12 \\ 16 \\ 29 \end{pmatrix}\), \(CB = \begin{pmatrix} 27 & 7 & 9 \end{pmatrix}\), and \(CA = \begin{pmatrix} 20 & 26 \end{pmatrix}\).

3. Find two matrices \(A, B \in M_{2 \times 2}(\mathbb{R})\) where \(AB\) is the zero matrix but \(BA\) is not.

Solution: Take \(A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}\), \(B = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}\). Then \(AB = 0\) but \(BA = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} \neq 0\).

4. Section 2.3 of [FIS], Problem 3.

Solution:

(a) We have \([U]_{\beta}^{\gamma} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & 1 \end{pmatrix}\), \([T]_{\beta} = \begin{pmatrix} 2 & 3 & 0 \\ 0 & 3 & 6 \\ 0 & 0 & 4 \end{pmatrix}\), and finally \([UT]_{\beta}^{\gamma} = \begin{pmatrix} 2 & 6 & 6 \\ 0 & 0 & 4 \\ 2 & 0 & -6 \end{pmatrix}\).

(b) We have \([h(x)]_{\beta} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}\) and \([U(h(x))]_{\gamma} = U_{\beta}^{\gamma}[h(x)]_{\beta} = \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix}\).
5. Section 2.3 of [FIS], Problem 4 (a), (b).

Solution:

(a) \[ T(A)_{\alpha} = \begin{pmatrix} 1 \\ -1 \\ 4 \\ 6 \end{pmatrix} \].

(b) \[ T(f(x))_{\alpha} = \begin{pmatrix} -6 \\ 2 \\ 0 \\ 6 \end{pmatrix} \].

6. Section 2.4 of [FIS], Problem 1.

Solution:

(a) False. It should be \((T^\beta_\alpha)^{-1} = T^{-1}_\beta\)

(b) True. See Appendix B.

(c) False. \(L_A\) can only map \(R^n\) to \(R^m\).

(d) False. It isomorphic to \(R^6\).

(e) True. This is because \(\dim(P_n(F)) = n + 1\).

(f) False. We have \(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = I\) but \(A\) and \(B\) are not invertible since they are not square.

(g) True. Since we have both \(A\) and \((A^{-1})^{-1}\) are the inverse of \(A^{-1}\), by the uniqueness of inverse we can conclude that they are the same.

(h) True. We have that \(L_{A^{-1}}\) would be the inverse of \(L_A\).

(i) True. This is the definition.

7. Suppose \(A\) and \(B\) are invertible \(n \times n\) matrices.

(a) Prove that \((AB)^{-1} = B^{-1}A^{-1}\).

(b) Prove that \((A^t)^{-1} = (A^{-1})^t\).

Solution:

(a) We have \((AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AA^{-1} = I\) and \((B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}B = I\).

(b) Since \(AA^{-1} = A^{-1}A = I\), taking transpose we have \((AA^{-1})^t = (A^{-1}A)^t = I\) or \((A^{-1})^tA^t = A^t(A^{-1})^t = I\). Thus \((A^t)^{-1} = (A^{-1})^t\).

8. Prove Theorem 2.21 of [FIS]. This shows that any finite dimensional vector space \(V\) of dimension \(n\) is isomorphic to \(R^n\).

Solution:

9. (a) Let \(A\) and \(B\) be \(n \times n\) matrices such that \(AB\) is invertible. Prove that both \(A\) and \(B\) are invertible.

(b) Give an example of two noninvertible matrices whose product is invertible.

(c) Prove or give a counterexample: If \(A\) and \(B\) are nonzero \(n \times n\) matrices with \(AB\) the zero matrix then \(A\) is not invertible.

Solution:

(a) Note that, \(\text{rank}(AB) \leq \text{rank}(A) \leq n\) as \(\text{ColSp}(AB) \subseteq \text{ColSp}(A) \subseteq R^n\). Now \(AB\) invertible, implies \(\text{rank}(AB) = n\), thus \(\text{rank}(A) = n\) and \(A\) is invertible. Since \(B = A^{-1}(AB)\), \(B\) is also invertible.

(b) The matrices must be non-square. Take \(A = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\). Then \(AB = (1)\), which is invertible.
(c) If $A$ and $B$ are nonzero $n \times n$ matrices with $AB$ the zero matrix then $A$ is not invertible. If $A$ is invertible, then $A^{-1}$ exists. Then $AB = 0$ implies that $B = A^{-1}(AB) = 0$, contradiction!

10. Find the inverse of the following matrix, and check your answer two different ways.

$$A = \begin{pmatrix} 2 & 1 & 2 \\ 1 & -1 & 0 \\ 4 & -2 & 1 \end{pmatrix}.$$

**Solution:** We have to find RREF for the super-augmented matrix $[A | I] = \begin{pmatrix} 2 & 1 & 2 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 \\ 4 & -2 & 1 & 0 & 0 & 1 \end{pmatrix}$ and we get $A^{-1} = \begin{pmatrix} -1 & -5 & 2 \\ -1 & -6 & 2 \\ 2 & 8 & -3 \end{pmatrix}$. One can directly check that $AA^{-1} = I$. 

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