Homework 3

MATH 416: Abstract Linear Algebra

Due date: February 14, 2018

Textbooks: In the assignment, the two texts are abbreviated as follows:


1. Section 1.6 of [FIS], Problem 1.

Solution: (a) No. The empty set is its basis.
(b) Yes. This is the result of the Replacement Theorem.
(c) No. For example, the set of all polynomials has no finite basis.
(d) No. \( \mathbb{R}^2 \) has \( \{(1,0), (1,1)\} \) and \( \{(1,0), (0,1)\} \) as bases.
(e) Yes. This is the Corollary after Replacement Theorem.
(f) No. It’s \( n+1 \).
(g) No. It’s \( m \times n \).
(h) Yes. This is the Replacement Theorem.
(i) No. For \( S = \{1,2\} \), a subset of \( \mathbb{R} \), then \( 5 = 1 \times 1 + 2 \times 2 = 3 \times 1 + 1 \times 2 \).
(j) Yes. This is Theorem 1.11 in [FIS].
(k) Yes. It’s \( \{0\} \) and \( V \) respectively.
(l) Yes. This is the Corollary 2 after Replacement Theorem.

2. Section 1.6 of [FIS], Problem 2, parts (a) and (b). Determine which of the following sets are bases for \( \mathbb{R}^3 \).

(a) \( \{(1,0,-1),(2,5,1),(0,-4,3)\} \)
(b) \( \{(2,-4,1),(0,3,-1),(6,0,-1)\} \)

Solution: It’s enough to check there are 3 vectors and the set is linear independent.

(a) Yes as we have
\[
\begin{pmatrix}
1 & 2 & 0 & 0 \\
0 & 5 & -4 & 0 \\
-1 & 1 & 3 & 0
\end{pmatrix}
\equiv
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{pmatrix}.
\]

(b) No as we have
\[
\begin{pmatrix}
2 & 0 & 6 & 0 \\
-4 & 3 & 0 & 0 \\
1 & -1 & -1 & 0
\end{pmatrix}
\equiv
\begin{pmatrix}
1 & 0 & 3 & 0 \\
0 & 1 & 4 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]

3. Section 1.6 of [FIS], Problem 8.

Solution: To solve this kind of questions, we can write the vectors into a matrix as below and do the Gauss-Jordan elimination with no swapping allowed,

\[
\begin{pmatrix}
2 & -3 & 4 & -5 & 2 \\
-6 & 9 & -12 & 15 & -6 \\
3 & -2 & 7 & -9 & 1 \\
2 & -8 & 2 & -2 & 6 \\
-1 & 1 & 2 & 1 & -3 \\
0 & -3 & -18 & 9 & 12 \\
1 & 0 & -2 & 3 & -2 \\
1 & -1 & 1 & -9 & 7
\end{pmatrix}
\equiv
\begin{pmatrix}
0 & -3 & 8 & -11 & 6 \\
0 & 0 & 0 & 0 & 0 \\
0 & -2 & 13 & -18 & 7 \\
0 & -8 & 6 & -8 & 10 \\
0 & 1 & 0 & 4 & -5 \\
0 & 1 & 6 & -3 & -4 \\
1 & 0 & -2 & 3 & -2 \\
0 & -1 & 5 & -15 & 11
\end{pmatrix}
\]
\[
\begin{pmatrix}
0 & 0 & 26 & -20 & 6 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 4 & -5 \\
0 & 0 & 54 & -32 & -22 \\
0 & 0 & -6 & -7 & -1 \\
0 & 1 & 6 & -3 & -4 \\
1 & 0 & -2 & 3 & -2 \\
0 & 0 & 11 & -18 & 7
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

and the row with all entries 0 can be omitted. So \{u_1, u_3, u_6, u_7\} would be the basis for \(W\) (the answer here will not be unique, for example one can replace \(u_1\) by \(u_4, u_5\) or \(u_8\)).

4. Recall from HW1.3 that, the subset \(U\) of all upper triangular matrices in \(M_{n \times n}(\mathbb{R})\) forms a subspace. Find a basis for \(U\) and use it to compute the dimension of \(U\).

**Solution:** For \(1 \leq i \leq j \leq n\), let \(A_{ij}\) be the matrix with \((i, j)\)-th entry one and all other entries being zero.

We claim that, \(S = \{A_{ij} \mid 1 \leq i \leq j \leq n\}\) is linearly independent. This follows from the fact that, \(\sum_{1 \leq i \leq j \leq n} c_{ij} A_{ij} = 0\) implies that

\[
\begin{pmatrix}
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0
\end{pmatrix}
= \begin{pmatrix}
c_{11} & c_{12} & \cdots & c_{1n} \\
c_{22} & c_{23} & \cdots & c_{2n} \\
0 & c_{33} & \cdots & c_{3n} \\
0 & 0 & \cdots & c_{4n} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & c_{nn}
\end{pmatrix}
\begin{pmatrix}
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0
\end{pmatrix}
\]

and thus \(c_{ij} = 0\) for all \(1 \leq i \leq j \leq n\).

We also claim that, \(\text{span}(S) = U\). This follows from the fact, that for \(B \in U\) with \(B = ((b_{ij}))_{i,j=1}^{n}\) where \(b_{ij} = 0\) for \(i < j\), we have \(B = \sum_{1 \leq i \leq j \leq n} b_{ij} A_{ij}\).

Thus \(S\) is a basis for \(U\) and \(\dim(U) = |S| = n(n + 1)/2\).

5. Suppose \(W\) is a subspace of a finite-dimensional vector space \(V\). For some \(v \in V\) not in \(W\), set \(X = \text{span}(W \cup \{v\})\). Prove that \(\dim(X) = \dim(W) + 1\).

**Solution:** By hypothesis, \(W\) is a subspace of a finite-dimensional vector space \(V\) and \(X = \text{span}(W \cup \{v\})\) for some \(v \in V \setminus W\). Thus there exists a finite basis \(\beta = \{u_1, u_2, \ldots, u_k\}\) of \(W\) where \(k = \dim(W) < \infty\). We will show that, \(S = \{v\} \cup \beta\) is a basis for \(X\) and this completes the proof since \(|S| = k + 1\).

First we prove that, \(S\) is linearly independent. Suppose that,

\[a_0 v + a_1 u_1 + a_2 u_2 + \cdots + a_k u_k = 0.\]
Then $a_0 = 0$, otherwise $v = \frac{a_1}{a_0}u_1 - \frac{a_2}{a_0}u_2 - \cdots - \frac{a_k}{a_0}u_k \in W$, contradiction. Thus, $a_1u_1 + a_2u_2 + \cdots + a_ku_k = 0$. Since, $\beta = \{u_1, u_2, \ldots, u_k\}$ is linearly independent, we have $a_1 = a_2 = \cdots = a_k = 0$.

Now we prove that, $S$ spans $X = \text{span}(W \cup \{v\})$. Any vector $w \in X$ can be written as $w = av + bu$ for some $a,b \in \mathbb{R}, u \in W$. But, $u \in W = \text{span}(\beta)$. Thus $w \in \text{span}(\{v\} \cup \beta) = \text{span}(S)$.

Hence, $S$ is a basis for $X$.

6. Section 1.6 of [FIS], Problem 20.

Solution: (a) If $S = \emptyset$ or $S = \{0\}$, then we have $V = \{0\}$ and the empty set can generate $V$. Otherwise we can choose a nonzero vector $u_1 \in S$, and continuing pick $u_{k+1}$ such that $u_{k+1} \notin \text{span}(\{u_1, u_2, \ldots, u_k\})$. The process would terminate before $k > n$ otherwise we can find linearly independent set with size more than $n$, which violates the conclusion of the replacement theorem. If it terminates at $k = n$, then we know the set is the desired basis. If it terminates at $k < n$, then this means we cannot find any vector to be the vector $u_{k+1}$. So any vectors in $S$ is a linear combination of $\beta = \{u_1, u_2, \ldots, u_k\}$ and hence $\beta$ can generate $V$ since $S$ can. But by Replacement Theorem we have $n \leq k$. This is impossible.

(b) If $S$ has less than $n$ vectors, the process must terminate at $k < n$. It’s impossible.