Homework 10
MATH 416: Abstract Linear Algebra
Due date: May 2, 2018

Textbooks: In the assignment, the two texts are abbreviated as follows:


1. Section 6.4 of [FIS], Problem 2 parts (a), (c), and (e).
2. Section 6.4 of [FIS], Problem 4.
3. Let \( T \) be a normal operator on a finite-dimensional inner product space \( V \).
   (a) Prove that \( \mathcal{N}(T) = \mathcal{N}(T^*) \) and \( \mathcal{R}(T) = \mathcal{R}(T^*) \).
   (b) Prove that the subspaces \( \mathcal{N}(T) \) and \( \mathcal{R}(T) \) are orthogonal.
   (c) Give an example of a (non-normal) linear operator \( S \) where \( \mathcal{N}(S) \neq \mathcal{N}(S^*) \) and \( \mathcal{R}(S) \neq \mathcal{R}(S^*) \).
   \textbf{Hint:} Problem 6.3.12 is your friend here.
4. A matrix \( A \in \mathcal{M}_{n \times n}(\mathbb{R}) \) is \textit{Gramian} if there is a \( B \in \mathcal{M}_{n \times n}(\mathbb{R}) \) such that \( A = B^t B \). Prove that \( A \) is Gramian if and only if \( A \) is symmetric and all of its eigenvalues are non-negative.
   \textbf{Hint:} For \( (\Leftarrow) \), note that \( A \) is diagonalizable via an orthonormal basis \( \{u_1, \ldots, u_n\} \) where \( u_i \) is an eigenvector of \( A \) with eigenvalue \( \lambda_i \). Consider the linear operator \( T \) on \( \mathbb{R}^n \) where \( T(u_i) = \sqrt{\lambda_i} u_i \). Now take \( B = [T]_{\text{std}} \) and check that \( A = B^t B \).
5. Suppose that \( v_1, \ldots, v_n \) are vectors in \( \mathbb{R}^n \) and let \( P \) be the parallelepiped spanned by them. Consider the matrix \( G \in \mathcal{M}_{n \times n}(\mathbb{R}) \) where \( G_{ij} = \langle v_i, v_j \rangle \). (As usual, the inner product here is just the ordinary dot product.)
   (a) Show that \( G \) is Gramian.
   (b) Show that \( \det(G) \geq 0 \).
   (c) Show that the unsigned volume of \( P \) is \( \sqrt{\det(G)} \).
   In fact, \( G \) is usually called the Gram matrix of these vectors.
6. Section 6.5 of [FIS], Problem 11.
7. Section 6.5 of [FIS], Problem 17.