Homework 0

MATH 416: Abstract Linear Algebra

Due date: Do Not Submit

1. Given a vector $\mathbf{u}$ in a vector space $V$ over $\mathbb{R}$, show that there is a unique vector $\mathbf{w}$ such that $\mathbf{u} + \mathbf{w} = \mathbf{0}$.

Solution: Given a vector $\mathbf{u} \in V$, suppose there are two vectors $\mathbf{w}, \mathbf{v}$ such that $\mathbf{u} + \mathbf{w} = \mathbf{0}, \mathbf{u} + \mathbf{v} = \mathbf{0}$. It is enough to show that $\mathbf{w} = \mathbf{v}$, which will imply that there exists a unique additive inverse. We have

$$\mathbf{w} + \mathbf{u} = \mathbf{u} + \mathbf{w} = \mathbf{0} = \mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}.$$

By cancellation law for vector addition, we have $\mathbf{w} = \mathbf{v}$. This completes the proof.

2. Let $V$ denote the set of all $m \times n$ matrices with real entries with the matrix addition and scalar multiplication given by component-wise addition and multiplication. Prove that, $V$ is a vector space over $\mathbb{R}$.

Solution: We have

$$V = \{ A = ((a_{ij}))_{1 \leq i \leq m; 1 \leq j \leq n} \mid a_{ij} \in \mathbb{R} \text{ for } 1 \leq i \leq m; 1 \leq j \leq n \}$$

with vector addition given by component-wise addition and scalar multiplication given by component-wise scalar multiplication. We check the eight axioms of vector space.

1. For all $A = ((a_{ij}))_{1 \leq i \leq m; 1 \leq j \leq n}, B = ((b_{ij}))_{1 \leq i \leq m; 1 \leq j \leq n} \in V$, we have

$$A + B = ((a_{ij}))_{1 \leq i \leq m; 1 \leq j \leq n} + ((b_{ij}))_{1 \leq i \leq m; 1 \leq j \leq n}
= ((a_{ij} + b_{ij}))_{1 \leq i \leq m; 1 \leq j \leq n}
= ((b_{ij} + a_{ij}))_{1 \leq i \leq m; 1 \leq j \leq n}
= ((b_{ij}))_{1 \leq i \leq m; 1 \leq j \leq n} + ((a_{ij}))_{1 \leq i \leq m; 1 \leq j \leq n}
= B + A.$$

2. Associativity follows from component-wise associativity.

3. The zero vector is the zero matrix $\mathbf{0} = ((0))_{1 \leq i \leq m; 1 \leq j \leq n}$ with all entries equal to 0 such that for $A = ((a_{ij}))_{1 \leq i \leq m; 1 \leq j \leq n} \in V$ we have $A + \mathbf{0} = A$.

4. For $A = ((a_{ij}))_{1 \leq i \leq m; 1 \leq j \leq n} \in V$, the matrix $B = ((-a_{ij}))_{1 \leq i \leq m; 1 \leq j \leq n}$ satisfies $A + B = \mathbf{0}$.

5. For $A = ((a_{ij}))_{1 \leq i \leq m; 1 \leq j \leq n} \in V$, we have $1 \cdot A = 1 \cdot ((a_{ij}))_{1 \leq i \leq m; 1 \leq j \leq n} = ((1 \cdot a_{ij}))_{1 \leq i \leq m; 1 \leq j \leq n} = A$.

6. Associativity of scalar multiplication follows from component-wise associativity for scalar multiplication.

7. For $c \in \mathbb{R}, A = ((a_{ij}))_{1 \leq i \leq m; 1 \leq j \leq n} \in V, B = ((b_{ij}))_{1 \leq i \leq m; 1 \leq j \leq n} \in V$ we have

$$c \cdot (A + B) = c \cdot ((a_{ij} + b_{ij}))_{1 \leq i \leq m; 1 \leq j \leq n}
= ((ca_{ij} + cb_{ij}))_{1 \leq i \leq m; 1 \leq j \leq n}
= ((ca_{ij}))_{1 \leq i \leq m; 1 \leq j \leq n} + ((cb_{ij}))_{1 \leq i \leq m; 1 \leq j \leq n} = c \cdot A + c \cdot B.$$

8. For $c, d \in \mathbb{R}, A = ((a_{ij}))_{1 \leq i \leq m; 1 \leq j \leq n} \in V$ we have

$$(c + d) \cdot A = (c + d) \cdot ((a_{ij}))_{1 \leq i \leq m; 1 \leq j \leq n}
= ((ca_{ij} + da_{ij}))_{1 \leq i \leq m; 1 \leq j \leq n}
= ((ca_{ij}))_{1 \leq i \leq m; 1 \leq j \leq n} + ((da_{ij}))_{1 \leq i \leq m; 1 \leq j \leq n} = c \cdot A + d \cdot A.$$

Thus $V$ is a vector space under matrix addition and component-wise scalar multiplication.
3. For any scalar \( a \) in \( \mathbb{R} \), show that

\[
a \cdot 0 = 0.
\]

**Solution:** We take a scalar \( a \in \mathbb{R} \). We have

\[
a \cdot 0 + a \cdot 0 = a \cdot (0 + 0) = a \cdot 0 = a \cdot 0 + 0 = 0 + a \cdot 0,
\]

where we used vector space axiom 7), 3), 3), 1) in the successive equalities, respectively. Now, using cancellation law for vector addition, we have \( a \cdot 0 = 0 \). This completes the proof.