Problem 1. Let $\| \cdot \|_J$ denote the norm of James space $J$. For every $F \subset \mathbb{N}$ with $2 \leq \#F < \infty$, if $F = \{p_1 < \cdots < p_n\}$ define for every $x \in c_0$ 

$$
\|x\|_F = \left( \sum_{i=1}^{n-1} (x(p_i) - x(p_{i+1}))^2 \right)^{1/2}
$$

and then define $\|x\|_J = \sup \{ \|x\|_F : F \subset \mathbb{N}, 2 \leq \#F < \infty \}$. 

Prove that for every $x \in c_0$ we have $\|x\|_J \leq \|x\|_J \leq \sqrt{2} \|x\|_J$.

Problem 2. A sequence $(x_k)$ in a Banach space with a Schauder basis $(e_i)$ is called a skipped block sequence if for every $k$ we have $\max \text{supp}(x_k) + 1 < \min \text{supp}(x_{k+1})$. Prove that for any skipped block sequence $(x_k)_{k=1}^n$ in $J$ we have 

$$
\left\| \sum_{k=1}^{n} x_k \right\|_J \geq \frac{1}{\sqrt{2}} \left( \sum_{k=1}^{n} \|x_k\|_J^2 \right)^{1/2}.
$$

Deduce that every closed infinite dimensional subspace of $J$ contains an isomorphic copy of $\ell_2$.

Problem 3. Let $1 \leq p < \infty$ and $f \in L_p$. The support of $f$ is defined as the smallest closed set $F$ so that $1_F \cdot f = f$ $\lambda$-a.e. (It is well known that such a set exists and you may take it for granted).

(i) Let $(f_n)$ be a normalized sequence in $L_p$ with $\text{supp}(f_n) \cap \text{supp}(f_m) = \emptyset$ for all $n \neq m$. Prove that it is isometrically equivalent to the unit vector basis of $\ell_p$.

(ii) Let $(f_n)$ be a normalized sequence in $L_p$ so that $\lim_n \lambda(\text{supp}(f_n)) = 0$. Prove that $(f_n)$ has a subsequence that is equivalent to the unit vector basis of $\ell_p$.

*Hint: for all $f \in L_p$ and $\varepsilon > 0$ there exists $\delta > 0$ so that whenever $\lambda(A) < \delta$ then $\int_A |f|^p d\lambda < \varepsilon$. 