Proposition: Let $x^{**} \in J^{**}$.

i) $\lim x^{**}(e^*_n)$ exists.

ii) $x^{**} \in J$ if and only if $\lim_{n} x^{**}(e^*_n) = 0$.

Proof: (i) Assume not, i.e. $a = \liminf_{n} x^{**}(e^*_n) < \limsup_{n} x^{**}(e^*_n) = b$.

By taking $x^{**} = x^{**} - aS^{**}$ we may assume $0 = a < b$.

Thus, there are $p_1 < q_1 < p_2 < q_2 < \ldots < p_k < q_k < \ldots$

s.t. $\forall k \in \mathbb{N}, x^{**}(e^*_{p_k}) > \frac{2b}{3}$ \& $x^{**}(e^*_{q_k}) < \frac{b}{3}$

For $N \in \mathbb{N}$ take $F = \{p_1 < q_1 < \ldots < p_N < q_N\}$

and calculate $2\|\sum_{i=1}^{q_N} x^{**}(e^*_i)e_i\|_F^2 \geq \sum_{i=1}^{N-1} (x^{**}(e^*_{p_i}) - x^{**}(e^*_{q_i}))^2 \geq (N-1)\frac{b^2}{9}$. Thus

$\|x^{**}\|^2 \geq \|\sum_{i=1}^{q_N} x^{**}(e^*_i)e_i\|_F^2 > (N-1)\frac{b^2}{18} \forall N \in \mathbb{N}$

which is absurd.

ii) If $x^{**} \in J$ then $x^{**} = \sum_{i=1}^{\infty} x^{**}(e^*_i)\hat{e}_i$ (in norm), thus $\lim_{n} x^{**}(e^*_n) = 0$.

If, on the other hand, $\lim_{n} x^{**}(e^*_n) = 0$ we will show that $\sum_{i=1}^{\infty} x^{**}(e^*_i)e_i$ is Cauchy in $\|\cdot\|_1$. If not, there exists $\varepsilon > 0$ & $p_1 < q_1 < p_2 < q_2 < \ldots < p_k < q_k$ s.t. $\forall k \in \mathbb{N}$ $\|\sum_{i=1}^{p_k} x^{**}(e^*_i)e_i\| > \varepsilon$.

Thus, $\forall k \in \mathbb{N} \exists F_k \in \mathcal{N}$ finite s.t. $\|\sum_{i=\hat{F}_k}^{q_k} x^{**}(e^*_i)e_i\| > \varepsilon$. 
Thus, if $F_k = \{ r_1^k < \ldots < r_{q_k}^k \}$ then if $x_k = \frac{g_k}{i=1} x_k^*(e_i) e_i$, 

$$\sum_{i=1}^{k-1} \left( x_k(r_i^k) - x_k(r_{i+1}^k) \right)^2 + \left( x_k(r_k^k) - x_k(r_1^k) \right)^2 \geq 2 \varepsilon^2$$

If we set $G_k = \{ r \in F_k : p_k \leq r \leq q_k \} = \{ r_1^k < \ldots < r_{q_k}^k \}$ 

Then $\sum_{i=1}^{k-1} \left( x_k(r_i^k) - x_k(r_{i+1}^k) \right)^2 \geq 2 \varepsilon^2 - 4 \varepsilon^2 \| x_k \|_{\infty}^2$

Because $\lim_n x^*(e_n^*) = 0$, \exists $k_0 \in \mathbb{N}$ s.t. $\forall k > k_0 \| x_k \|_{\infty} < \frac{\varepsilon^2}{8\varepsilon}$

Take $m = N + k_0 + N$ and define $F = \bigcup_{k=k_0+1}^{k_0+N} G_k$.

Then, $\| x^* \|_F \geq \| \sum_{i=1}^{q_k} x^*(e_i^*) e_i \|_F \geq \| \sum_{i=1}^{q_k^k} x^*(e_i^*) e_i \|_F \\

\frac{1}{\sqrt{2}} \left( \sum_{k=k_0+1}^{k_0+N} \sum_{i=1}^{q_k} \left( x_k(r_i^k) - x_k(r_{i+1}^k) \right)^2 \right)^{1/2} \geq \frac{N (\varepsilon^2)^{1/2}}{\sqrt{2}} = \varepsilon \frac{N}{2}$

$\Rightarrow \| x^* \|_F \geq \varepsilon \frac{N}{2} \quad \forall N \in \mathbb{N}$ which is absurd.

It remains to prove that $J$ is isometrically isomorphic to $J$.

**Proposition**: The map $T : J \rightarrow J^{**}$ given by 

$T x = -x(\omega) s_{**} + \sum_{n=1}^{\infty} x(n) h_1 \hat{e}_n$ is a well defined linear isometry that is onto.
Proof: We show that it is well defined and an isometry. It suffices to show that $\| Tx \| = \| x \|$. (Note, $Tx$ is well defined for $x \in C_0$).

\[
\| Tx \| = \| -x(1) S^* + \sum_{i=1}^{m'} x(i1) \hat{e}_i \| \\
= \lim_{m \to \infty} \left\| \sum_{i=1}^{m} (x(i1) - x(1)) e_i + \sum_{i=n}^{m} x(i1) e_i \right\|.
\]

Fix $m \geq n+1$ and take $F = \{ p_1, \ldots, p_k \}$. We will find $G$ s.t. $\| x \|_F = \| y_m \|_G$. Wlog, we may assume $p_k \leq n+1$.

Case 1: $2 \leq p_k$. Define $G = \{ p_1, \ldots, p_k, p_{k-1} \}$. Then,

\[
\| y_m \|_G^2 = \sum_{i=1}^{k-1} \| (x_{p_i} - x(1)) - (x_{p_{i+1}} - x(1)) \|^2 + \| (x_{p_k} - x(1)) - (x_{p_1} - x(1)) \|^2
\]

Fix $m \geq n+1$ and take $F = \{ p_1, \ldots, p_k \}$. We will find $G$ s.t. $\| x \|_F = \| y_m \|_G$. Wlog, we may assume $p_k \leq n+1$.

Case 2: $p_k = 1$. Define $G = \{ p_2^{-1}, p_3^{-1}, \ldots, p_k^{-1}, n+1 \}$. A similar calculation yields $\sqrt{2} \| y_m \|_G^2 = \sqrt{2} \| x \|_F^2$.

Thus, $\| x \|_G \leq \| y_m \|_G \Rightarrow \| x \| \leq \| Tx \|$

In an analogous manner we prove $\| y_m \|_G \geq \| x \|_G$ thus $\| Tx \| = \| x \|$. i.e., $T$ is an isometry.

To see that $T$ is onto, it suffices to show that it has dense image. Indeed, $S^* = T_{p_2} \oplus \hat{e}_i = T_{p_1}, \forall i \in N$. Thus, $J^{\perp} = \langle S^* \rangle \oplus \hat{O}$ is in the image of $T$. 


Proposition: If J does not have property (u) and thus it does not embed in a space with an unconditional basis.

Proof: If $y_n = \sum_{i=1}^{\infty} e_i$, we proved that $(y_n)$ is u-Cauchy.

If J had property (u) there would exist a u-C uc $\Sigma_i x_i$

s.t. $(y_n - \frac{1}{n^2} x_i) \to 0$.

But $J^{**}$ is separable so $C_{00} \not\subseteq J$. By Banach-Steinhaus, the series $\Sigma_i x_i$ converges to some $x \in J$ (unconditionally). Thus, $y_n \to x \in J$ which is absurd.

$L_p$-spaces, $1 \leq p < \infty$

Given a measure space $(X, \mathcal{A}, \mu)$, i.e.,

- X is a non-empty set,
- $\mathcal{A}$ is a σ-algebra of sets on X,
- $\mu: \mathcal{A} \to [0, \infty]$ is a positive σ-additive measure.

We define $\forall 1 \leq p < \infty$

$$L_p(X, \mathcal{A}, \mu) = \{ f: X \to \mathbb{R} \text{ measurable s.t. } \int_X |f|^p d\mu < \infty \}$$

equipped with $\|f\|_p = (\int_X |f|^p d\mu)^{1/p}$.

To be more precise, the objects of $L_p(X, \mathcal{A}, \mu)$ are equivalence classes of $\mu$-a.e. equal functions.

Also, $L_{\infty}(X, \mathcal{A}, \mu) = \{ f: X \to \mathbb{R} \text{ measurable, } f \text{ is essentially bounded} \}$. 
**Example:** \( X = [0,1], \mathcal{O} = \mathcal{B}(C[0,1]) \) (Borel sets of \([0,1]\)), 
\( \mu = \lambda \) (Lebesgue measure on \([0,1]\)). We denote 
\( L^p[0,1] := L^p(C[0,1], \mathcal{B}(C[0,1]), \lambda) \)

We will focus on these spaces.

**Theorem:** Let \( X \) be a Polish space and \( \mu \) be a 6-finite positive measure on \( \mathcal{B}(X) \) (the Borel 6-algebra of \( X \)). Let \( 1 \leq p \leq \infty \).

- If \( \mu \) is \( \mu \)-atomic then \( L^p(X, \mathcal{B}(X), \mu) = L^p[0,1] \).
- If \( \mu \) is purely atomic then
  - if it has \( n \) atoms, \( n \in \mathbb{N} \), \( L^p(X, \mathcal{B}(X), \mu) = C^n \).
  - if it has infinite atoms, \( L^p(X, \mathcal{B}(X), \mu) = C^\infty \).
- If \( \mu \) is neither purely atomic, nor \( \mu \)-atomic then
  - if it has \( n \) atoms, \( n \in \mathbb{N} \), \( L^p(X, \mathcal{B}(X), \mu) = C^n \oplus L^p[0,1] \).
  - if it has infinite atoms, \( L^p(X, \mathcal{B}(X), \mu) = C^\infty \oplus L^p[0,1] \).

We will neither prove nor use this theorem, which generalizes the fact \( L^p(\mathbb{R}) = L^p[0,1] \). 

**Henceforth,** \( L^p := L^p[0,1] \).