Recall: If $X$ has an unconditional basis $(e_i)$ then it is boundedly complete if and only if $c_0$ does not embed into $X$.

**Proposition (James):** Let $X$ be a $B$-space with an unconditional basis $(e_i)$. TFAE

1. $(e_i)$ is not shrinking.
2. $e_i$ embeds into $X$.
3. $e_i$ embeds into $X$ as a separable subspace.
4. There exists a complemented block sequence in $X$ that is equivalent to the urv of $e_i$.

**Proof:** Remaining implication:

(i)⇒(iv). If $(e_i)$ is not shrinking then $(e_i^*)$ is not b.c.

In a block sequence $(x_k^*)$ in $H = C[a,b]$ that is equivalent to the urv of $c_0$. That is $f, B > 0$ s.t.

$$\forall a_1, \ldots, a_n \in \mathbb{R} \quad \frac{1}{A} \max_{i=1}^n |a_i| \leq \left\| \sum_{i=1}^n a_i x_i \right\| \leq B \max_{i=1}^n |a_i|.$$ 

Let $x_k \in B_X$ with $x_k^*(y_k) > \frac{1}{2} \|x_k\| > \frac{1}{2A}$.

If $x_k^* = \sum_{i=p_{k-1}^+}^{p_k^+} b_i e_i^*$ then $x_k = \left( \sum_{i=p_{k-1}^+}^{p_k^+} b_i x_k^* \right)$ and observe

$$\|x_k\| < 2AK \ (K \text{ is the unconditionality constant}) \forall K \in \mathbb{N},$$

$$x_k^*(x_k) = \frac{x_k^*(y_k)}{x_k^*(y_k)} = 1 \quad \text{and} \quad x_k^*(x_m) = d_{Kn}. \text{ We show that } (x_k)_K$$

is the desired sequence. First, for $a_1, \ldots, a_n \in \mathbb{R}$

$$\left\| \sum_{i=1}^n a_i x_i \right\| \leq \sum_{i=1}^n |a_i| \|x_i\| \leq 2AK \sum_{i=1}^n |a_i|.$$
For the lower inequality take
\[ b_i = \text{sgn}(a_i) \] and set
\[ x^k = \sum_{i=1}^{n} b_i x_i^k. \] Then
\[ \|x^k\| \leq B \max \{|b_i|\} = B. \]
Thus,
\[ \|x^k\| \sum_{i=1}^{n} a_i |x_i^k| \geq \sum_{i=1}^{n} a_i x_i^k(x_i) = \sum_{i=1}^{n} a_i x_i^k(x_i) \]
\[ = \sum_{i=1}^{n} a_i b_i = \sum_{i=1}^{n} |a_i| \|x_i^k\|. \] Thus,
\[ \|x^k\| \geq \frac{1}{B} \sum_{i=1}^{n} |a_i| \|x_i^k\|. \]
Unlike \( C_0 \), we need to find the projection onto \((x_i)\).
We define \( P \) by \( P = \sum_{k=1}^{\infty} x_k^k(x_k) x_k \).

If we show that \( P \) is well defined \& bounded it will be the desired projection. We show that it is bounded on \( \langle e_i : i \in \mathbb{N} \rangle \).
Let \( x = \sum_{i=1}^{n} a_i e_i \).

Then,
\[ \|P x\| = \left\| \sum_{k=1}^{\infty} x_k^k(x_k) x_k \right\| \leq 2AK \sum_{k=1}^{\infty} |x_k^k(x_k)| \]
\[ = 2AK \left| \sum_{k=1}^{\infty} \text{sgn}(x_k^k(x_k)) x_k^k(x_k) \right| = 2AK \left( \sum_{k=1}^{\infty} b_k x_k^k(x_k) \right) \]
\[ = b_k \]
\[ \leq 2AK \sum_{k=1}^{\infty} b_k \|x_k^k\| \|x_k^k\| \leq 2AKB \max \{|b_k|\} \|x\| = \square \]

We want to define a separable Banach space \( J \) with the properties:

i) \( J \) does not contain \( C_0 \)

ii) \( J \) does not contain \( l^2 \)

iii) \( J \) is isometrically isomorphic to \( J^{**} \)
      (in particular, \( J^{**} \) is separable).

iv) \( J \) is of cotype-1 in \( J^{**} \). (In particular, \( J \) is non-
      reflexive. (and thus, \( J \) does not have an unconditional basis)

v) \( J \) does not embed in a subspace with an unconditional basis.
To deduce some of these properties we first need to introduce another notion.

**Recall:** A sequence \((x_k)_k\) in a B-space \(X\) is called \(w\)-Cauchy
if \(\forall x \in X^*\), \(\lim_{n \to \infty} x(x_k)\) exists.

A series \(\sum_k x_k\) in \(X\) is called \(w\)-UC if \(\forall x \in X^*\), \(\sum_k |x(x_k)| < \infty\).

**Remark:** if \(\sum_k x_k\) is \(w\)-UC then \(\left( \sum_k x_k \right)_n\) is \(w\)-Cauchy.

**Def:** A B-space \(X\) has Pelczyński's property (\(\alpha\))
if for every \(w\)-Cauchy sequence \((y_k)_k\) in \(X\) \(\exists\) a \(w\)-UC series \(\sum_k x_k\) so that
\[
\lim_{n \to \infty} (y_n - \sum_{k=1}^{n} x_k) = 0.
\]

We want to prove the following:

**Proposition 1:** If \(X\) has an unconditional basis then \(X\) has property (\(\alpha\)).

**Proposition 2:** If \(X\) has property (\(\alpha\)) and \(Y\) is a closed subspace of \(X\) then \(Y\) has property (\(\alpha\)).

**Remark:** Prop 2 is not entirely obvious.

To prove prop 1 we will use the following homework exercise: if \((x_k)_k\) is a bounded block sequence in a space with an unconditional basis \(B\) \((x_k)_k\)

is \(w\)-null then \((x_k)_k\) has a subsequence equivalent to the urb of \(B\).
Remark: If \((x_k)_k\) is a sequence in a Banach space that has a subseq. equivalent to the uwb of \(\ell_p\) then \((x_k)_k\) is not w-Cauchy. Indeed, \((x_k)_k\) is w-Cauchy then so are all of its subsequences. Thus, the uwb of \(\ell_p\) is w-Cauchy which is absurd.

Proof of Prop. 1: Let \(X\) have a K-unconditional basis \((e_i)\), & let \((x_k)_k\) be a w-Cauchy sequence in \(X\).

Then, \(\forall i \in \mathbb{N}\) \(a_i = \lim_{k \to \infty} e_i^* (x_k)\) exists & for all \(n \in \mathbb{N}\)
\[ \| \sum_{i=1}^{n} a_i e_i \| = \lim_{k \to \infty} \sum_{i=1}^{n} e_i^* (x_k) e_i \leq \lim_{k \to \infty} \| S_n (x_k) \| \leq K \sup \| x_k \| \]
We will show that \(\sum a_i e_i\) is wuc & \(X_n - \sum_{i=1}^{n} a_i e_i \xrightarrow{w^*} 0\).

For the first statement, let \(x^* \in X^*\). Then, for \(n \in \mathbb{N}\),
\[ \sum_{i=1}^{n} | x^* (a_i e_i) | = \sum_{i=1}^{n} \frac{\| S_n (x_k) \|}{K} \leq \frac{\sup \| x_k \|}{K} \]
\[ \leq \| x^* \| K \sup \| x_k \| \].

Thus, \(\sum a_i e_i\) is wuc.

In particular, \((x_n - \sum_{i=1}^{n} a_i e_i)_n\) is w-Cauchy.

For the second part, observe that \(\forall i \in \mathbb{N}\)
\[ e_i^* (x_n - \sum_{i=1}^{n} a_i e_i) = e_i^* (x_n) - a_i \to 0 \]
Towards a contradiction, assume \(x_n - \sum_{i=1}^{n} a_i e_i \xrightarrow{w} 0\).

Then, \(\exists k \in \mathbb{N}\), \(x^* X^*\) & a subseq. s.t. \(x^* (x_{n_k} - \sum_{i=1}^{n_k} a_i e_i) > 2^{-k}\)
\(\forall k \in \mathbb{N}\).

Thus, this \((x_{n_k})\) has no w-null subsequence.
By a gliding hump argument, there is a further subsequence \((w_{km})_m\) of \((w_k)\) that is orthogonal to a block sequence \((z_m)_m\). But \((w_{km})\) is not w-null i.e. neither is \((z_m)\) and as we observed \((z_m)\) is not w-Cauchy, i.e., neither is \((w_{km})\). This is absurd.

Exercise: Let \(X\) be a reflexive Banach space with an \(S\)-basis. Prove that \(X\) has property \((u)\).