Def: A series $\sum x_n$ in a Banach space is called weakly unconditionally convergent if $\forall x^* \in X^*$, $\sum_{n=1}^{\infty} x^*(x_n) < \infty$.

Proposition: Let $\sum x_n$ be a series in a Banach space. TFAE
i) $\sum x_n$ is WUC
ii) $\exists C$ s.t. $\forall FCH$ finite, $\| \mathcal{F} \| \leq C$
iii) $\exists T \in \mathcal{L}(c_0, X)$ s.t. $T e_n = x_n \quad \forall n \in \mathbb{N}$.

Theorem (Bessaga-Pelczyński): Let $X$ be a Banach space. TFAE
i) $X$ does not contain an isomorphic copy of $c_0$.
ii) Every WUC series converges unconditionally in norm.

$\Rightarrow$ i) is obvious because in $c_0$, $\sum x_n$ is WUC but not convergent.

For $i) \Rightarrow ii)$ we need the following:

Theorem: Let $X$ be a Banach space & $T \in \mathcal{L}(c_0, X)$. TFAE
i) $T$ is compact (i.e. $\overline{\{(B_{c_0})^{\infty}, X \text{ compact}\}}$
ii) $T$ is weakly compact (i.e. $\overline{T(B_{c_0})^{\infty}, X \text{ w-compact}}$
iii) $T$ is strictly singular (i.e. $\not\exists V \subset c_0$ closed & infinite dim. s.t. $T|V$ is an embedding)
iv) If $x_n = \langle e_i, x \rangle_{c_0}$ then $\lim_n \|Tx_n\| = 0$. 

Proof of Bessaga-Pelczynski: \( i \Rightarrow ii \)

Assume \( \exists WUC \ X_n \) that does not converge unconditionally.

- \( X_n \) is WUC: \( T : c_0 \to X \) with \( T(a) = \sum_n a(n)X_n \) is bounded.
- \( \exists X_n \) does not converge unconditionally: \( \exists > 0 \) s.t. \( \forall n \in \mathbb{N} \) \( \exists F_n \subset \mathbb{N} \) finite with \( \min F_n > n \) \& \( \| \sum_{i \in F_n} x_i \| \geq \varepsilon \).

Define \( a_n \in B_{c_0} \): \( a_n(i) = \begin{cases} 1 & : i \in F_n \\ 0 & : \text{otherwise} \end{cases} \) & observe
\[
T a_n = \sum_{i \in F_n} x_i \Rightarrow \| T \|_\infty \geq \varepsilon \ \forall n \in \mathbb{N}.
\]
By the previous theorem \( T \) is not strictly singular, i.e., \( \exists Y \subset c_0 \) s.t. \( T_{|Y} : Y \to X \) is an isomorphism. But \( c_0 \lesssim Y \to X \Rightarrow c_0 \lesssim X \).

Proof of other theorem: \( i) \Rightarrow ii) \) is obvious.

For \( ii) \Rightarrow iii) \) Assume that \( T \) is w-compact but not strictly singular, i.e., \( \exists Y \subset c_0 \) inf. dim. s.t. \( T_{|Y} : Y \to T(Y) \) is an isomorphism. But \( \overline{T(B_Y)}^w \) is w-compact, then \( \overline{T(B_Y)}^w \) is w-compact (\( \overline{T(Y)} \to Y \) is w-conv.) & has non-empty interior, thus \( B_Y \) is w-compact \( \Rightarrow Y \) is reflexive. But \( c_0 \lesssim Y \) which is absurd. For \( iii) \Rightarrow iv) \) Assume that
\( \exists > 0 \) s.t. for \( n < n_2 < \ldots < n_k < \ldots \), \( \| T(y_k) \| \geq \varepsilon \ \forall k \in \mathbb{N} \).
Take $x_k \in Y_{n_{12}}$ with $\|x_k\| = 1$ s.t. $\|T x_k\| > 2$.

Observe that $\forall i \in I, \lim_{k \to \infty} e_i(x_k) = 0$. By the gliding hump argument, $(x_k)$ has a subsequence (called $(x_i)$) that is equivalent to $(e_i)$.

In particular, $(x_k) \Rightarrow 0$ implies $(T x_k) \Rightarrow 0$.

By passing to another subsequence, $(T x_k)$ is $S$-basic with some constant $k$.

**Claim:** $(x_i) = (T x_k)$, i.e., $\overline{T(x_i)}_{k \to \infty}$ is an isomorphism which would be absurd, because $T$ is strictly singular.

**Proof of claim:** Because $T$ bounded, $\forall i_0, \ldots, i_n \in \mathbb{R}

\| \sum_{i=1}^{n} a_i x_i \| \leq \| T \| \sum_{i=1}^{n} |a_i| \| x_i \|.

Because $(T x_k)$ is basic for some $k > 0$, let $C > 0$ s.t.

\[ \| \sum_{i=1}^{n} a_i x_i \| \geq C \| \sum_{i=1}^{n} |a_i| \| T x_i \|, \]

\[ \geq \frac{\varepsilon}{2kC} \| \sum_{i=1}^{n} a_i x_i \|. \]

Thus, $(x_i) \Rightarrow (T x_i)$.

**Remarks to prove (iv) i).** Set $T_n : C_0 \to X, T_n = T S_n$, where $S_n : C_0 \to C_0$ canonical projections. Each $T_n$ is finite rank, thus compact. Also, $T - T_n = T S_{(n \to m)}$ & $\| T S_{(n \to m)} \| = \text{sup} \{ \| T S_{(n \to m)} x \| : x \in B_{C_0} \} = \| T L_{n \to m} \| \to 0$. i.e. $T_n \rightarrow T$ thus $T$ is compact as a limit of compact operators in operator norm.
Remark: One can also define $w$-convergence of series. If $\Sigma x_n$ converges unconditionally, every subseries converges weakly.

Theorem (Orlicz-Pettis): Let $\Sigma x_n$ be a series in a Banach space $X$ such that any subseries converges weakly. Then, $\Sigma x_n$ converges unconditionally in norm.

The proof will be given later.

Remark: If a series satisfies the assumption of the above theorem then it must be WUC. Indeed, if $x \in X^*$

$$\sum_{n=1}^{\infty} x^*(x_n) = \sum_{n=1}^{\infty} x^*(x_n) + \sum_{n=1}^{\infty} x^*(x_n) < \infty.$$ 

Exercise: Find a series $\Sigma x_n$, in a Banach space $X$ all permutations of which converge weakly, but not all subseries of which converge weakly.

Hint: if $c_0 \subseteq X$ then such a series does not exist.